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The interpretation of forensic evidence using a likelihood ratio

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SUMMARY

Forensic scientists often have to interpret refractive index measurements made on fragments of window glass, some taken from the scene of a crime and some found on a suspect. Adopting a model used recently by Lindley (1977) and Seheult (1978) this contribution proposes a non-Bayesian solution based on a likelihood ratio. The problem of deciding whether the fragments have come from a common source is distinguished from the problem of deciding the guilt or innocence of the suspect.

Some key words: Error probability; Forensic science; Identification; Likelihood ratio; Maximum likelihood; Nuisance parameter.

1. INTRODUCTION

Lindley (1977) and Seheult (1978) have discussed the following problem in forensic science. A window is broken at the scene of a crime. Fragments of glass are found in the clothing of a suspect, and measurements are made of their refractive index. Measurements are also made on fragments from the broken window. What evidence concerning the guilt of the suspect do the measurements provide?

We adopt the assumptions and notation introduced by Lindley. Measurements are normally distributed with known variance σ^2 about a mean value θ, characteristic of the window from which the fragment was taken. The mean for the scene window is θ_1, and for the fragments from the suspect θ_2. The latter fragments are assumed to be window glass and, for the present, to come from a single source. The sample sizes for scene and suspect are m and n, and the sample means X and Y, respectively. The distribution of θ over all windows will be taken to be normal $N(\mu, \tau^2)$ with μ and τ known. This assumption is unrealistic but leads to a simple solution of which the implications can be easily understood. Lindley considers the hypotheses I, that the two sets of fragments come from a common source, and its complement $\bar{I}$. Given I, $\theta_1 = \theta_2$ is assumed to have the distribution of θ over all windows. Given $\bar{I}$, θ_1 and θ_2 are assigned this distribution independently. Lindley derives the likelihood ratio for I versus $\bar{I}$. Although his results are presented in a Bayesian context, he also gives a table of error probabilities for a decision rule of the form 'decide guilty if the likelihood ratio exceeds k', for a range of values of k.

Seheult points out that the distributions assigned to θ_1 and θ_2 are potentially based on objective information and suggests that they be formally incorporated into I and $\bar{I}$, which become simple hypotheses. Lindley's rule then provides the most powerful test, in the usual Neyman–Pearson sense, of I versus $\bar{I}$, and thus need not be regarded as characteristically Bayesian. An unstated consequence of Seheult's argument is that Lindley's Table 2 can be read as a selection of values of (α, β), where α is the size and $1-\beta$ the power of the test.

The present contribution has two purposes. The first is to argue the case for, and examine the consequences of, treating θ_1 and θ_2 asymmetrically, given $\bar{I}$. This approach leads to a decision rule which is in general different from Lindley's, although in an important special case their properties are approximately the same. The second purpose is to examine the relationship between evidence concerning I and evidence concerning G, the event that the suspect is guilty. This clarifies the role played by the assumption of a single source for Y. It also shows that none of the approaches so far suggested, including the present one, can be regarded as solving the problem of deciding between G and $\bar{G}$.

2. Identity and nonidentity

In discussing the respective roles of θ_1 and θ_2 we start from the premise that an orthodox non-Bayesian approach to a statistical problem requires that we do not attribute a probability distribution to an unknown parameter without special justification. In the present example, provided the person is suspected for reasons not connected with the discovery of the particular set of fragments, and no information is offered as to their source, $\bar{I}$ can be interpreted as saying that the fragments came to be present 'by chance', entailing a random choice of value for θ_2. There is no corresponding argument for regarding the value of θ_1 as randomly chosen.

This argument leads us to regard θ_1 as a nuisance parameter, which in the Bayesian approach is integrated out by reference to a prior distribution. There are several other ways in which θ_1 could be eliminated. We could consider $p(Y \mid \theta_1)/p(Y)$, where

$$p(Y) = \int p(Y \mid \theta_2)\, p(\theta_2)\, d\theta_2,$$

and use X, a 'reliable' estimate of θ_1 under both hypotheses, to estimate θ_1. Or, denoting the observations from the scene by $\{x_i\}$ $(i = 1, \ldots, m)$, those from the suspect by $\{y_j\}$ $(j = 1, \ldots, n)$ and the overall mean by W, we could consider a ratio of marginal likelihoods, suitable ancillary statistics being $\{x_i - W\}$ $(i = 1, \ldots, m)$ and $\{y_j - W\}$ $(j = 1, \ldots, n)$ under I, $\{x_i - X\}$ and $\{y_j\}$ under $\bar{I}$. Since under each hypothesis the ancillaries are independent of the sufficient statistic for θ_1, these marginal likelihoods are also conditional likelihoods.

An alternative approach, which we pursue here, is to consider $p(X, Y \mid I, \theta_1)/p(X, Y \mid \bar{I}, \theta_1)$, where $p(X, Y \mid I, \theta_1) = p(X \mid \theta_1) p(Y \mid \theta_1)$ and $p(X, Y \mid \bar{I}, \theta_1) = p(X \mid \theta_1) p(Y)$, and to replace θ_1 by its maximum likelihood estimate under each hypothesis, i.e. by W and X respectively. The resulting likelihood ratio is

$$\frac{\sigma_1 \sigma_2}{a \sigma \sigma_3} \exp\left\{-\frac{(X-Y)^2}{2a^2\sigma^2} + \frac{(Y-\mu)^2}{2\sigma_2^2}\right\}, \tag{1}$$

where $\sigma_1^2 = \sigma^2/m$, $\sigma_2^2 = \tau^2 + \sigma^2/n$, $\sigma_3^2 = \sigma^2/(m+n)$ and $a^2 = 1/m + 1/n$.

Lindley assumes, realistically, that τ is much larger than σ. For further simplification we can take $m = n$, and absorb the sample size into the definition of σ. These assumptions lead to the following approximate form of (1)

$$\frac{\tau}{\sigma} \exp\left\{-\frac{(X-Y)^2}{4\sigma^2} + \frac{(Y-\mu)^2}{2\tau^2}\right\}. \tag{2}$$

This is very similar in form to Lindley's expression

$$\frac{\tau}{\sigma\sqrt{2}} \exp\left\{-\frac{(X-Y)^2}{4\sigma^2} + \frac{(Z-\mu)^2}{2\tau^2}\right\}, \tag{3}$$

where $Z = \frac{1}{2}(X + Y)$. Each contains a term which corresponds to the usual test of the hypothesis that $\theta_1 = \theta_2$ against a general alternative, together with a term which increases the likelihood ratio in favour of identity when the observed refractive indices are unusual.

It turns out that when (2) and (3) are used as the basis of decision rules, their properties are approximately the same. In the spirit of our earlier discussion we show this for fixed θ_1.

We assume (2) to be used in the form: decide I if (2) exceeds K, while (3) is used in the form: decide I if (3) exceeds k. Given $\bar{I}$, and taking $K = k\surd 2$, an error is made if

$$-\tfrac{1}{4}\tau^2\xi^2\sigma^{-2}+\tfrac{1}{8}\eta^2 > \log(k\sigma\surd 2/\tau), \tag{4}$$

where ξ is approximately $N(\delta, 1)$, $\delta = (\theta_1 - \mu)/\tau$ and η is approximately $N(0, 4)$ for (2) and $N(\delta, 1)$ for (3). To a good approximation η can be neglected in both cases.

Given I, an error is made if

$$-\tfrac{1}{2}\xi^2+\tfrac{1}{2}\eta^2 \leqslant \log(k\sigma\surd 2/\tau), \tag{5}$$

where ξ is $N(0, 1)$ and η is $N(\delta, \sigma^2/\tau^2)$ for (2) and $N(\delta, \tfrac{1}{2}\sigma^2/\tau^2)$ for (3). In this instance η cannot be neglected for all θ_1, but as an alternative we can replace $\tfrac{1}{2}\eta^2$ by its expectation. The neglecting of terms in σ^2/τ^2 leads to the same expression in both cases.

In the light of this approximate equivalence we can interpret the values in Lindley's Table 2 as approximate average error probabilities for a rule based on (2), the average being with respect to the distribution of θ_1 in the population of all windows. This average may or may not be of practical interest: it is possible that windows which tend to be broken in the course of a crime have a distribution of θ_1 very different from that in the whole population. If we look at the problem for a given value of θ_1 it is of interest to know the maximum values taken by the error probabilities as θ_1 varies. Using (5), the maximum probability of wrongly deciding $\bar{I}$ occurs when $\theta_1 = \mu$. If we use (4), and neglect η^2, the maximum probability of wrongly deciding I also occurs when $\theta_1 = \mu$.

These approximate maximum error probabilities are given in Table 1 for the range of k examined by Lindley. We take $\tau/\sigma = 100$ and 200; the latter case can be thought of as corresponding to a more diffuse distribution of refractive index over all windows, or to a more precise method of determining refractive index, or to having more fragments in both samples. These values are roughly 50% larger than the corresponding 'average' values given by Lindley.

Table 1. *Approximate maximum error probabilities for a decision rule based on* (2) *or* (3)

τ/σ		$k = 1$	$k = 2$	$k = 4$	$k = 8$	$k = 16$	$k = 32$	$k = 64$	$k = 128$
100	pr (decide $\bar{I}$ | I)	0·003	0·008	0·017	0·037	0·085	0·208	0·655	1·0
	pr (decide I | $\bar{I}$)	0·033	0·030	0·027	0·024	0·020	0·014	0·005	0·000
200	pr (decide $\bar{I}$ | I)	0·002	0·003	0·008	0·017	0·037	0·085	0·208	0·655
	pr (decide I | $\bar{I}$)	0·018	0·017	0·015	0·014	0·012	0·010	0·007	0·003

3. Guilty or not guilty?

We are of course only concerned with the evidence about I in so far as it has a bearing on the guilt or innocence of the suspect. The real quantity of interest to the Bayesian is $p(X, Y \mid G)/p(X, Y \mid \bar{G})$ and from the point of view of § 2 is

$$\max_{\theta_1} p(X, Y \mid G, \theta_1)/\max_{\theta_1} p(X, Y \mid \bar{G}, \theta_1).$$

We define the event T, that fragments were transferred from the broken window to the suspect and persisted there until discovery by the police, and A, that the suspect came into contact with glass from some other source. We can write $p(X, Y \mid G)$ as

$$p(X, Y \mid T, \bar{A}, G)\,p(T, \bar{A} \mid G) + p(X, Y \mid T, A, G)\,p(T, A \mid G) + p(X, Y \mid \bar{T}, A, G)\,p(\bar{T}, A \mid G), \tag{6}$$

with a similar expression for $p(X, Y|\bar{G})$. Assuming that $p(A|G) = p(A|\bar{G}) = p_A$, say, $p(T|G) = p_T$, A and T are independent given G and that $p(T|\bar{G})$ is effectively zero, we have that

$$\frac{p(X, Y|G)}{p(X, Y|\bar{G})} = \frac{p(X, Y|T, \bar{A})}{p(X, Y|\bar{T}, A)} \frac{(1-p_A)p_T}{p_A} + \frac{p(X, Y|T, A)p_T}{p(X, Y|\bar{T}, A)} + (1-p_T). \quad (7)$$

There is no reason to suppose that the second term in (7) can in general be ignored. We may, though, be prepared to make our judgements about G on the assumption that S, the event that fragments from a single source are found on the suspect, has occurred. Now

$$\frac{p(G|X, Y, S)}{p(\bar{G}|X, Y, S)} = \frac{p(X, Y, S|G)}{p(X, Y, S|\bar{G})} \frac{p(G)}{p(\bar{G})} \quad (8)$$

and since $S = (T \cap \bar{A}) \cup (\bar{T} \cap A)$, $p(X, Y, S|G)$ is given by omitting the second term from (6) and the likelihood ratio by omitting the second term from (7). Here $T \cap \bar{A}$ and $\bar{T} \cap A$ specify the same conditions as I and $\bar{I}$ respectively, and although they are not complementary in the wider context of this section, it is useful to retain Lindley's notation; we can then write

$$\frac{p(X, Y, S|G)}{p(X, Y, S|\bar{G})} = 1 + p_T\left\{(p_A^{-1} - 1)\frac{p(X, Y|I)}{p(X, Y|\bar{I})} - 1\right\}. \quad (9)$$

Thus the likelihood ratio for I versus $\bar{I}$ need not give even an approximate indication of the value of (9). D. W. Smith, in an unpublished M.Sc. dissertation, examined the following data arising from a case in the Birmingham area:

Fragments from scene window:	1·518255, 1·518199, 1·518273, 1·518292, 1·518273, 1·518292, 1·518255, 1·518392.
Fragments found on suspect:	1·518273, 1·518347, 1·518119, 1·518218, 1·518283, 1·518338, 1·518246, 1·518255.

Using a method equivalent to Lindley's, but with a 'prior' distribution based on data collected by the Home Office Central Research Establishment, he obtained the value 116·5 for $p(X, Y|I)/p(X, Y|\bar{I})$. Even if we fix p_A at 0·01 say, the value of (9) can vary from about 1, if p_T is very small, to over 10^4, if p_T is close to 1. It seems likely that realistic estimates of p_T and p_A would be difficult to obtain in practice. A similar problem will of course occur in the evaluation of

$$\max_{\theta_1} p(X, Y, S|G, \theta_1) / \max_{\theta_1} p(X, Y, S|\bar{G}, \theta_1).$$

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