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A BRIEF REVIEW OF THE DEVELOPMENT OF QUALITATIVE CONTROL THEORY IN BELARUS

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The main stages of the development of the qualitative control and observation theory in Belarus are analyzed. Some fundamental results are presented. Unsolved problems are formulated.

Keywords: *control theory, controllability, observability, stabilization, modal control, reconstruction, canonical forms.*

INTRODUCTION

Mathematical control theory (MCT) is a modern section of general mathematical systems theory, which is based on the concepts of a state, input actions (inputs), and output responses (outputs) of the system. According to this theory, a system can be specified by means of a so-called black box, more precisely, by a transition input–state–output mapping with an internal description, if the mapping of transition of states is given (i.e., if the law of transition of states is given and, therefore, there is a possibility to look inside the "black box"), and by an external description, when only an input–output mapping is given. If the input mapping turns out to be surjective in the internal description, then the system is considered fully controllable. If the output mapping is injective, the system is called fully observable. If control (or observation) is a time process, then the system is called dynamic (or discrete if time intervals are isolated). For a dynamic system, input effects can be divided into external (selected, as a rule, from a definite class, which is a class of controls) and internal (characterizing the starting states of the system, which are initial data, frequently fixed). Each control problem includes the following characteristic features:

1) equations of motion; these are relations (usually functional differential) between the input and output variables and state variables thus specifying an input–output mapping,

2) a constraint on the phase trajectory, which is a constraint on the state variables,

3) a constraint on the control (description of the class of admissible controls), and

4) the objective of control, which may be either quantitative or qualitative.

In the first case, we deal with extremization (minimization or maximization) on a set of admissible controls of a quantitative index of control performance, i.e., a quality criterion (functional). These are optimal control (optimization) problems.

In the second case, such a quantitative index is absent, as a rule, and the problem refers to qualitative control and observation theory (QCOT). Studies on control theory oriented toward the development of constructive algorithms for deriving desired controls with regard for the capabilities of modern computers have resulted by now in a constructive control theory. Certainly, such a classification of control problems is conditional to a certain extent.

One of the major sections of modern QCOT, being actively developed in Belarus, is QCOT for dynamic systems. Its sources are in classical control theory and the theory of stability of motion. In this context, it is necessary to mention an important role of Academician E. A. Barbashin in organizing and propagandizing the new theory in Belarus. He promoted moving to Minsk in 60s of Prof. F. M. Kirillova and Prof. R. Gabasov, who fruitfully worked in control theory, had a profound impact on the formation and development of mathematical control theory (MCT) in Belarus, and managed to create in a comparatively short period an original Belarussian scientific school on control theory, which is now one of the leaders recognized by the global scientific community. Belarussian experts in control theory obtained basic results in many directions

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of QCOT, in particular, in control and observation systems with aftereffect. These results establish international priority of the Belarussian scientific school. Let us present some statistical data. While 276 of 1330 studies on QCOT for finite-dimensional systems [1] are published in Russian and only 46 of them belong to Belarussian authors, already 228 publications have Belarussian citizenship among 395 Russian-language ones out of 815 studies on infinite-dimensional systems for the same period [2]. Belarussian specialists in QCOT participated in many international conferences, symposia, and congresses as invited lecturers and directed their sections (in particular, at the 11th (1990) and 12th (1996) IFAK Congresses).

Certainly, it is impossible to describe in detail all the publications on QCOT in Belarus because of the lack of space, and the reference list at the end of the paper is incomplete. Therefore, we only analyze below the main directions and the results obtained (indicating whenever possible mainly the authors and references for Ph.D. and Dr.Sci. theses) on QCOT in Belarus. The history of this problem till 1983 is adequately presented in the reviews [3–6] and bibliography [1, 2].

Two main trends can be seen in the development of QCOT in Belarus:

(i) focusing of the studies on improvement and generalization of main problem statements and investigation of new and, as a rule, deeper properties of finite-dimensional controlled and observable systems, and

(ii) considering more complex control and observation systems such as systems of functional differential equations with aftereffect, systems of equations with partial derivatives, infinite-dimensional systems, hybrid systems, etc.

1. INVESTIGATION INTO THE QUALITATIVE CHARACTERISTICS OF FINITE-DIMENSIONAL CONTROLLABLE AND OBSERVABLE SYSTEMS

Of studies in this subject area, we will first point out the studies immediately generalizing Kalman's classical results on controllability and observability. Let, for example, the symbol $x(t, t_0, x_0, u)$ mean the state at a time $t > t_0$ of a finite-dimensional dynamic system with the initial state $x(t_0) = x_0 \in R^n$ at the initial moment $t = t_0$ and the control $u = u(t)$, $t > t_0$. Then for given matrices $H \in R^{m\times n}$ and $M \in R^{n\times k}$ of the subspaces $\{x \in R^n : Hx = 0\}$ and $\{x \in R^n : x = My\}$, the system is recognized as (R. Gabasov, F. M. Kirillova):

(i) relatively controllable if there is an instant of time $t_* > t_0$ for which a piecewise-continuous control $u(t), t \in [t_0, t_*]$, such that $Hx(t, t_0, x_0, u) = 0$ can be found for any initial state $x_0 \in R^n$,

(ii) conditionally controllable if there exists an instant of time $t_* > t_0$ for which a piecewise-continuous control $u(t)$, $t \in [t_0, t_*]$, can be found such that $x(t_*, t_0, x_0, u) = 0$ for any initial state $x_0 = My_0$ $(\forall y_0 \in R^k)$.

Efficient parametric criteria for both conditional and relative controllability are obtained in [7] for linear stationary systems. Constraints on the set of initial and finite states can be combined in one problem: a system is called suffciently MH-conditionally-relatively controllable if time instants $t_* > t_0$ and piecewise-continuous control $u(t)$, $t \in [t_0, t_*]$, such that $Hx(t, t_0, x_0, u) \equiv 0, u(t) \equiv 0, t \geq t_*$, can be found for any initial state $x_0 = My_0$ $(\forall y_0 \in R^k)$.

An efficient parametric criterion of such controllability (V. M. Marchenko and V. L. Merezha) and dual problems of conditional and relative observability (R. Gabasov, R. M. Zhevnyak, F. M. Kirillova, and T. B. Kopeikina) can be found in [1–4, 8] for linear stationary systems. Note that the property of sufficient conditional-relative controllability is invariant as to whether the finite time instant $t_* = t(x_0)$ in the definition is considered dependent on the initial data as distinct from the definition of relative controllability, where this assumption, as examples show, is significant [3].

A number of studies on QCOT depend on the "minimum" properties of controllable dynamic systems, in particular, on controllability in special (elementary) classes of functions, such as relay controls and algebraic and trigonometric polynomials, on solution of linear differential equations (B. Sh. Shklyar) and, more generally in the class of outputs of dynamic controllers (V. V. Ignatenko) and in the class of Chebyshev functions (A. I. Astrovskii). The general result of these studies can be formulated as follows: the condition of controllability in the class of piecewise-continuous controls provides controllability of ordinary linear stationary systems in considerably narrower classes of functions. Dual results of observability are presented by V. V. Mulyarchik. R .Gabasov, F. M. Kirillova [7], and T. N. Gurina (Antonovich) [1–4] have clarified conditions of controllability in the class of positive controls. The problem on the minimum number of inputs for linear stationary systems was considered by R. Gabasov [7] with the following result: the minimum number of inputs of a controllable system is equal to the number of nontrivial (not equal identically to unity) invariant polynomials of the matrix specifying the structure of the proper dynamics of the system. B. Sh. Shklyar and V. M. Marchenko generalized this result to relative controllability and systems with constraints on the structure of the input device. The latter has proposed also a general method for calculation of the minimum number of inputs for various classes of dynamic systems.

The effect of linear feedback on the spectrum of linear stationary systems (in other words, the problem of modal control, control of a spectrum) was treated by I. K. Asmykovich and V. I. Bulatov. They have described, in particular, invariant (inaccessible) eigenvalues under the effect of various forms of feedback. The results obtained by L. E. Zabello [1, 24] on control by Lyapunov's index for nonstationary systems should be referred to this subject. A further generalization of the control problem for spectrum of a system is reconstruction of its dynamics when the choice of linear feedback controls not only the eigenvalues but also the structure of elementary divisors of the matrix specifying the proper dynamics of the system. These studies have been described by V. I. Yanovich.

Completing analysis of studies relevant to the first subject area, we should mention the studies [1–4, 7] in the field of controllability and observability of discrete and quantized systems (R. Gabasov, F. M. Kirillova, V. V. Krakhotko, S. A. Minyuk, R. F. Naumovich), game problems of controllability (V. M. Marchenko), controllability and observability of nonstationary systems (A. I. Astrovskii, L. E. Zabello, T. B. Kopeikina, etc.), controllability of variable-structure systems (B. S. Kalitin), variational approaches to controllability and observability problems (B. Sh. Mordukhovich), controllability and observability of nonlinear systems (R. Gabasov, S. Ya. Gorohovik, F. M. Kirillova, T. B. Kopeikina, etc.) mainly from the point of view of controllability and observability of their linear approximations. At present, so-called descriptor systems, i.e., systems that are not solved for the derivative, has came to be studied intensively. QCOT for such systems are studied by I. K. Asmykovich, V. I. Bulatov, V. V. Ignatenko, V. V. Krakhotko, V. M. Marchenko, O. N. Poddubnaya, T. S. Trofimchuk (Kalyuzhnaya), etc. We will mention also studies by A. I. Astrovskii and S. K. Korzhenevich on description of information sets in problems of observation with "fuzzy" noise.

2. INVESTIGATION INTO THE QUALITATIVE CHARACTERISTICS OF CONTROLLABLE AND OBSERVABLE INFINITE-DIMENSIONAL SYSTEMS

In characterizing the second tendency in the development of QCOT in Belarus, fundamental studies of qualitative properties of control and observation systems with aftereffect, described by functional differential equations should be mentioned. The results obtained in this direction should be considered separately since due to them the Belarussian mathematical sch firstool on control theory has got international recognition, and we will discuss them later.

Of other areas, we will mention the original developments of Prof. Yu. K. Lando and his disciples (V. T. Borukhova, V. K. Boiko, Yu. A. Bykadorov, G. I. Kabak, etc.) [1–4], who successfully applied the principle of conjugate correspondence, formulated by Yu. K. Lando, in the theory of normal boundary-value problems with control to studying QCOT in systems of integro-differential equations. In particular, they have formulated controllability criteria for various classes of such systems.

Basic results on QCOT for systems with multidimensional time are obtained by Academician I. V. Gaishun and his disciples (V. V. Goryachkin, Huag Van Kuang). In particular, for linear nondegenerate discrete two-parameter systems, he has found criteria of controllability and observability in classes of arbitrary and bounded sequences, has studied the control problem for a spectrum for such systems, etc. These results allowed him to develop operator methods for studying QCOT for Rosser' systems, widely used in processing multidimensional information arrays.

The investigations of V. T. Borukhov are devoted to various structural characteristics of dynamic systems given by the input–state–output mapping; in particular, he has obtained reversibility criteria and ways of constructing inverse systems for regular classes of linear lumped and distributed systems, has shown the relationships between inverse problems of mathematical physics and structural characteristics such as controllability, observability, reversibility, realization, and has proposed an approach to studying such characteristics based on binary linear relations.

There are individual publications [2–5] on QCOT for infinite-dimensional systems such as systems of equations with partial derivatives and systems in Banach spaces (A. I. Astrovskii, R. F. Naumovich, Ya. V. Radyno, etc.). However, it is too early to speak about comprehensive results in this area, since many QCOT problems for these systems lead to delicate questions of functional analysis and the theory of functions such as closedness, a basis property of systems of functions in function spaces, interpolation in a class of integer functions of a finite degree with nodes in roots of transcendental equations, etc. To avoid these difficulties, it is necessary to impose additional substantial requirements on the parameters of the control and observation systems being considered, for example, to require that the solutions of such systems can be expanded in the eigenvectors (and rooted vectors (functions)) of the respective operators specifying the proper dynamics of the system, etc.

3. CONTROL AND OBSERVATION SYSTEMS WITH AFTEREFFECT

Among the most important are the results of studies of Belarussian scientists in QCOT for dynamic control and observation systems described by functional differential equations (FDEs) with aftereffect of both delay and neutral type (for both lumped and distributed delay). The beginning of studies in this area is dated by N. N. Krasovskii's report to the 2nd IFAC Congress in 1963, where the problem of total controllability of a system with delayed argument was formulated. By and large, the Sverdlovsk school on control theory, headed by the Academician N. N. Krasovskii, has prepared experts in this area such as Academicians A. B. Kurzhanskii and Yu. S. Osipov, Profs. R. Gabasov and F. M. Kirillova, and other scientists, is basic for QCOT in systems with delay. This school possesses global priority in formulation of main problems of QCOT such as total controllability (N. N. Krasovskii, A. B. Kurzhanskii (1963, 1966)), stabilization by integral feedback (N. N. Krasovskii, Yu. S. Osipov (1963, 1965)), relative controllability (F. M. Kirillova, S. V.Churakova (1967) and, independently, L. Weiss (1967, 1970)), etc. Starting in the second half of 60s, the center of studies on QCOT in systems with aftereffect gradually moves to Minsk. The main result in these years is, mainly, the relative controllability of such systems. We will observe the main stages in the development of QCOT in Belarus for systems with aftereffect using, as an example, an elementary system:

with the delayed argument:

$$\dot{x}(t) = Ax(t) + A_1 x(t-h) + Bu(t), \quad t > t_0, \tag{1}$$

($A \in R^{n\times n}$, $A_1 \in R^{n\times n}$, $B \in R^{n\times r}$, $0 < h = \text{const}$ is the delay),

and with the initial conditions:

$$x(t_0 + \tau) = \varphi(\tau), \tau \in [-h, 0), x(t_0 + 0) = \varphi_0 \tag{2}$$

(here $\varphi = \varphi(\cdot)$ is a function, for example, continuous on $[-h, 0]$, of given class Ω).

System (1) is called relatively t_1-controllable [5, 7], where $t_1 > t_0$ if for any initial data φ, φ_0 from (2) and any n-vector $x_1 \in R^n$, there exists a piecewise-continuous control $u = u(t)$, $t \in [t_0, t_1]$, for which the corresponding solution $x(t) = x(t, t_0, \varphi_0, \varphi, u)$, $t > t_0$, of system (1) possesses the property $x(t_1) = x_1$. If we put in this definition $x_1 = 0 \in R^n$, then we obtain the problem of relative zero-controllability. Unlike Kalman systems ($A_1 = 0$), these problems, as examples show, are not equivalent to systems with aftereffect, which was not fully taken into account in the first studies on this subject (see [5, 7]). The problem of relative controllability was efficiently and comprehensively solved within the framework of the method of constitutive equation developed by F. M. Kirillova and R. Gabasov [2–5, 7, 8]. It turned out that system (1) is relatively t_1-controllable if and only if

$$\text{rank}\,[X_k(jh);\ k = 0, 1, \ldots, n-1;\ j = 0, 1, \ldots, \alpha] = n, \tag{3}$$

where $\alpha = \lim_{\varepsilon \to +0} \left[\frac{t_1 - \varepsilon}{h} \right]$, the symbol $[a]$ means the integer part of the number a, $X_k(t)$, $k = 0, 1, \ldots, n-1$, $t \geq 0$, is the solution of the constitutive equation corresponding to system (1)

$$X_{k+1}(t) = AX_k(t) + A_1 X_k(t-h),\ t \geq 0,\ k = 0, 1, 2, \ldots, \tag{4}$$

with the initial conditions $X_0(0) = B$, $X_0(t) = 0$ if $t \neq 0$. Criterion (3) of relative controllability was further generalized to systems with many delays and systems of neutral type (R. Gabasov, V. V. Krakhotko). However, the role of the constitutive equation in studying the properties of systems with delay is not investigated completely. It is not exhausted by its role in studying the problem of relative controllability but reflects definite intrinsic properties [5] of systems with delay; in particular, the fundamental matrix $F(t)$ of solutions of the corresponding homogeneous open-loop system (1) can be expressed by the formula (B. Sh. Shklyar)

$$F(t) = \sum_{i=0}^{+\infty} \sum_{j=0}^{p} X_i^1(jh)) \frac{(t - jh)^i}{i!},\ t \in [ph, (p+1)h], \quad p = 0, 1, 2, \ldots,$$

where $X_i^1(t)$ is the solution of Eq. (4) for $B = I_n$. Here, I_n is a unit $(n \times n)$-matrix that allows us to present the solution of system (1) as a series in the solutions of the constitutive equation

$$x(t) = \nu(t, \varphi_0, \varphi) + \sum_{k=0}^{+\infty} \sum_{t-jh>0} \frac{X_k(jh) \int_0^{t-jh} (t-\tau-jh)^k}{k!} u(\tau)d\tau, \ t > 0 \tag{5}$$

where the function $\nu(\cdot, \cdot, \cdot)$ depends only on initial data. Directly, such a representation refining the well-known Bellman and Coock representation [9] of a solution as contour integrals, is proposed by the author of this paper based on the algebraic properties of the solutions of the constitutive equations obtained by him. We will mention some of them:

1) the main identity:

$$(A + mA_1)^k B = \sum_{j=0}^{k} m^j X_k(jh), \ m \in K,$$

where K is an arbitrary ring whose elements commute with the matrices A, A_1, and B, in particular, K is a field of complex numbers;

2) the generalized Hamilton–Cayley theorem: solution of the constitutive equation satisfies its characteristic equation, i.e.,

$$\sum_{i=0}^{n} \sum_{j=0}^{i} r_{ij} X_{m-i}^1((\gamma - j)h) = 0, \text{ for } m = n, n+1, n+2, \ldots; \gamma = 0, 1, 2, \ldots$$

where r_{ij}, $j = 0, 1, \ldots, i$; $i = 0, 1, \ldots, n$, are the coefficients of the characteristic equation

$$\det[\lambda I_n - A - A_1 \exp(-\lambda h)] = \sum_{i=0}^{n} \sum_{j=0}^{i} r_{ij} \lambda^{n-i} \exp(-\lambda jh) = 0, \ \lambda \in C, \tag{6}$$

($r_{00} = 1$, C is a field of complex numbers)

of the open-loop system (1) (the Hamilton–Cayley theorem, well-known for matrices, follows herefrom for $A_1 = 0$). The results formulated allow us to select a finite number of generatrices in the linear hull of columns of the solution of the constitutive equation and, thus, promote the description of the set of relatively controlled (more precisely, relatively accessible) states of system (1).

Analysis of the problem of relative zero-controllability in systems with delay is significantly complicated by the fact that they can degenerate pointwise, i.e., at some instant of time $t_1 > t_0$, i.e., all the possible solutions of the corresponding open-loop system may not fill the whole space R^n. Systems being not pointwise degenerate are called pointwise complete. In the case of pointwise complete sets, the requirement of relative controllability and relative zero-controllability are equivalent. This fact holds also for two-dimensional ($n = 2$) linear stationary systems with aftereffect of general form since they are pointwise complete, but it ceases to be valid for systems with distributed delay and of neutral type with concentrated delay already for ($n = 3$). It is well known (V. V. Karpuk) that the equivalence of the concepts of relative controllability and relative zero-controllability is preserved for systems (1) with $n \le 5$ and is violated for $n = 10$. A problem on the maximum dimension of n for which properties of relative and relative zero-controllability of system (1) are equivalent remains open to date. Parametric conditions of relative zero-controllability of systems with a delayed argument, which may compete with criterion (3) in efficiency and completeness of the form, are yet to be found. Further results in this area, as well as for pointwise (and functional) completeness and controllability for initial function can be found in works of L. E. Zabello, V. V. Karpuk, T. B. Kopeikina, V. M. Marchenko, A. V. Metel'skii, S. A. Minnyuk, etc. [2–5, 8, 32].

R. Gabasov formulated problems of pointwise controllability as multipoint boundary-value problems for systems described by FDE with control. Two of them are analyzed by S. A. Minyuk: 1) the controllability at the points $\beta_0, \beta_1, \ldots, \beta_\gamma$, 2) the α-pointwise controllability. System (1) is called controlled at the points $\beta_0, \beta_1, \ldots, \beta_\gamma$ (strictly ordered in increasing order) if there exists a moment $t_1 > t_0 + \beta_\gamma$ such that for any initial data φ_0, φ and n-vectors $c_j, j = 0, 1, \ldots, \gamma$, there exists a piecewise-continuous control for which $x(t_1 - \beta_j, t_0, \varphi_0, \varphi, u) = c_j$; $j = 0, 1, \ldots, \gamma$. A system is considered α-pointwise controlled ($\alpha \ge 0$) if it is controlled at any points $\beta_0, \beta_1, \ldots, \beta_\gamma$ such that $0 = \beta_0 < \beta_1 < \ldots < \beta_\gamma \le \alpha$. The parametric criterion of solvability of the formulated problems can be found [2–5, 36, 37] by analogy with (3) with the use of the technique (and in terms) of constitutive equation, once again confirming its efficiency for studying finite-dimensional problems of controllability. The problem of pointwise controllability (α-pointwise controllability for any

$\alpha \geq 0$) was formulated and solved by V. M. Marchenko [2–5, 32]. He has proved that the property of α-pointwise controllability is saturated: if system (1) is α-pointwise controlled for $\alpha = \alpha_0 = (n-1)(n-2)h:2$, then it is also α-pointwise controlled for $\alpha \geq \alpha_0$, i.e., it is pointwise-controlled. Moreover, it is cleared up that system (1) is pointwise-controllable if and only if the one-parameter system

$$\dot{x}(t) = (A + mA_1)x(t) + Bu(t), t > t_0, \tag{7}$$

without delay is controlled in terms of Kalman for at least one real value of the parameter m. This result allows us to generalize to pointwise controllability of system (1) many positions of QCOT of the Kalman systems theory, in particular, to formulate dual problems of observability and to construct the theory of duality in problems of pointwise control and observation and to consider the problem of realization of systems with aftereffect. The results presented admit generalization to systems with a delayed argument of neutral type and to system with many delays [32].

One of the most difficult problems of QCOT in systems with aftereffect is the Krasovskii problem on complete damping in a finite time of a system with a delayed argument. R. Gabasov and F. M. Kirillova [7] have proposed a general scheme of analyzing this problem allowing us to specify for each specific system (1) the procedure of its check for complete controllability. They have attempted to develop further this scheme in order to use potentialities of modern computers (G. P. Razmyslovich), and to search (S. A. Minyuk, A. V. Metel'skii, N. N. Stepanyuk, etc.) for the conditions of solvability of the problem, directly expressed through the parameters of the system. Numerous attempts to obtain an explicit (parametric) criterion of complete controllability were undertaken also abroad. The parametric (spectral) criterion of complete controllability as the requirement [5]

$$\operatorname{rank}[\lambda I_n - A - A_1 \exp(-\lambda h), B] = n \quad \forall\, \lambda \in C, \tag{8}$$

was first expressed by V. M. Marchenko as a hypothesis in 1975 at a seminar on control processes (under the guidance of R. Gabasov and F. M. Kirillova) and was verified by him and S. A. Minyuk using an example of a two-dimensional ($n = 2$) system (1), and was confirmed in the general case in 1977 by V. M. Marchenko. Further results in this area can be found in [5, 8, 10]. Of interest is the fact [5, 32] that condition (8) cannot be transferred to the general case of the system of neutral type with a distributed delay (the number-theoretic nature of delays is significant here).

An important cycle of studies is fulfilled on feedback control theory. The problem of modal control (MC), having numerous applications, from which a number of priority results is obtained, occupies the central place here. We will begin with the formulation of the problem: the problem of control of a spectrum, traditional (W. M. Wonham, 1967) in Kalman systems, turns out to be unsuitable in systems with delay due to their infinite dimensions. Therefore, such systems were considered first with regard to their MC [2, 5] only in 1974 (V. I. Bulatov, T. S. Kalyuzhnaya, R. F. Naumovich). The problem of partial MC (control of any finite part of a spectrum) was formulated and analyzed; the approach was based on the Krasovskii and Osipov technique for studying the stabilization problem. I. K. Asmykovich and V. M. Marchenko (1976) have made a general formulation of the MC problem for system (1) [2, 5] as a problem of control of the coefficients of the characteristic equation (5) and, thus, an infinite-dimensional, in formulation, problem of spectrum control was substituted by a finite-dimensional one. To solve the problem, they proposed a linear feedback as difference controllers

$$u(t) = \sum_{j=0}^{\theta} Q_j x(t - jh), t > t_0, Q_j \in R^{r \times n} \ (j = 0, 1, \ldots, \theta). \tag{9}$$

It turned out that system (1) with a scalar input ($r = 1$) is modally controllable in the class of controllers (9) if and only if

$$\det W(m) = \det[B, (A + mA_1)B, \ldots, (A + mA_1)^{n-1}B] \equiv \text{const} \neq 0 \quad \forall m \in R. \tag{10}$$

The proof of this fact is based on the algebraic properties of the shift operator and the solutions of the constitutive equation. A similar result is obtained independently in the USA [2, 5] in terms of the theory of moduli (A. S. Morse, 1976) for a multiple-input system and a weakened formulation of the MC problem as a control of special chains of solutions of Eq. (5). The technique for solution of MC problem in the class of integral controllers with the use of the theory of integer functions of finite degree (in particular, the Wiener–Paley theorem) can be found in [5, 32], where the well-known Wonham theorem was generalized: system (1) is modally controllable (in the class of integral controllers) if and only if it is totally controllable. An analysis of a stabilization problem (in the class of difference controllers), which is a special case of a MC problem, as well as various generalizations of the MC problem to systems with many delays, of neutral type, with distributed delay in the case of both complete and incomplete information on the state (a generalization of the J. B. Pearson dynamic controller is constructed here), and an analysis of the problem of reconstruction of system dynamics can be found in [2, 5, 8, 32] in the works by I. K. Asmykovich, I. M. Borkovskaya, V. M. Marchenko, and V. I. Yanovich. Unfortunately, not all of the obtained results in this area have such a completed form as criterion (10).

Under the effect of the abstract approach to the construction of QCOT in the general system theory in the 70s and in the first half of the80s, the tendency is strengthened to the formulation and analysis of problems of controllability and observability in systems with delay based on the state-space method. The essence of the approach is as follows: a set $\Omega \ni \varphi$ of initial data in (2) with the joining condition $\varphi(0) = \varphi_0$ (or $(\varphi_0, \varphi) \in R^n \times \Omega$ otherwise) is identified with the space $\mathfrak{R}$ of the initial (and then current) states of the system. Then $\mathfrak{R}$-controllability (functional) can be interpreted as the admissible control, generating a trajectory connecting in a finite time two arbitrary given points from $\mathfrak{R}$ (for zero initial data, this is the problem of total accessibility). Similarly, the total observability is characterized as a possibility to use output measurements to distinguish initial data that have generated them. This topic, being extremely popular abroad, is actively developed in Belarus [2, 5, 37] by the R. Gabasov and F. M. Kirillova school (S. A. Minyuk, A. V. Metel'skii, B. Sh. Shklyar, etc.). It turned out that the problem of $\mathfrak{R}$-controllability can be solved only in extreme cases, even when $\mathfrak{R}$ is isomorphic to the Sobolev space $W_2^{(1)}([-h, 0], R^n)$. Therefore, the properties of controllability were considered then in a weaker sense: on the one hand, problems were studied, in which the trajectory should be put into an arbitrary neighborhood (in a topology of the space $\mathfrak{R}$) of a finite state (approximate controllability); on the other hand, the coincidence of the trajectory with a finite state was required on any interval with a length arbitrarily smaller than the value of delay. In this context, of special attention was the space $M_2 = R^n \times L_2([-h, 0], R^n)$. Let us formulate some results [2, 5, 37]: system (1) is M_2-approximately controllable if and only if the requirement $\det A_1 \neq 0$ is fulfilled, in addition, along with condition (8) (S. A. Minyuk, S. N. Lyakhovets, 1980); the necessary and sufficient condition of total observability of system (1) with the output $y(t) = Cx(t)$, $t > t_0$, consists in the requirement (S. A. Minyuk, A. V. Metel'skii (1978); B. Sh. Shklyar (1979)): $\det A_1 \neq 0$, $\operatorname{rank}[\lambda I_n - A' - A_1' \exp(-\lambda h), C'] = n \ \forall \lambda \in C$ (the prime means transposition). Thus, the whole class of sufficiently controllable and Kalman-observable systems (1) with $A_1 = 0$ is automatically eliminated from the functionally controllable and observable systems (at least in approximative sense). The reason is that the minimality of a state is not required, i.e., the states as the elements of a priori given topological space $\mathfrak{R}$ are, as a rule, not minimal. Historically, problems of functional controllability go back to the problem of E. A. Barbashin on the motion [10] along a given path (1960).

A new approach to the analysis of problems of functional controllability and observability is proposed by V. M. Marchenko [2, 5, 32] based on the concept of a minimum state (s-state). The essence of the approach is as follows: the set $R^n \times \Omega$ of the initial data, factorized on "adhesion" of corresponding solutions of the system for $t > t_0 + s$, is interpreted as a set ${}_sX_0$ of the initial s-states, and its image, by virtue of the system, gives a set of s-solutions and, at last, the contraction of this set (in the structure $R^n \times \Omega$) to the interval $[t - h, t]$ has a sense of the set ${}_sX_t$ of admissible s-states at the instant time $t > t_0$. Based on this, it is possible to construct a theory of controllability and observability of systems with aftereffect by analogy with the Kalman theory as controllability and observability of their s-states. Since it is difficult to operate with an s-state as a coset, a concept of s-information ${}_s\mathrm{I}_{t_0}(\varphi_0, \varphi, u))$ is introduced, for which s-states are sets of level; in particular, ${}_0\mathrm{I}_{t_0}(\varphi_0, \varphi, u) = {}_0\mathrm{I}_{t_0}(\varphi_0, \varphi, 0) = (\varphi_0, \gamma)$ for system (1), where $\gamma(\tau) = A_1\varphi(t_0 + \tau - h)$, $\tau \in [0, h]$. Thus, the space of s-states of system (1) for $A_1 = 0$ is finite-dimensional (is isomorphic to R^n). Introduction of s-information not only systematizes the accumulated experience in QCOT for systems with aftereffect but also leads to further generalizations, to problems of (s, t)-controllability and observability: a system is called (s, t)-controllable if for any initial data (φ_0, φ), $(\psi_0, \psi) \in R^n \times \Omega$ and admissible control $v(\tau)$, $\tau \in [t_1 - t, t_1]$, there exists an admissible control $u(\tau), \tau \in [t_1 - s, t_1]$, for which

$$x(t_1 + \tau, t_1 - s, \varphi_0, \varphi, u) = x(t_1 + \tau, t_1 - t, \psi_0, \psi, v), \ \tau \in [0, h] \tag{11}$$

(the problem of program pursuit of similar-type objects with discrimination of the evader). An object (system) is considered R^n-(s, t)-controllable if coincidence (11) of solutions in the previous definition is required only for $\tau = 0$. Of special interest are problems of R^n-(s, t)-controllability for $t = 0$ (relative controllability) and $t = s$ (relative zero-controllability), and (s, t)-controllability in terms of Krasovskii in time s. For linear nonstationary systems with aftereffect of general form (with distributed delay of general form), an immediate generalization of the Kalman theory of duality between controllability and observability is constructed. It should pointed out that for a system in differential form with joining condition, a system in integral form without joining condition turns out to be dual and vice versa. Details of the approach can be found in [32], where parametric criteria of controllability and observability in stationary case are obtained. Here, of special interest is the space ${}_sX_0$ for $s \overset{def}{=} \infty$ (the space of weak states). It is possible to show that all such spaces are isomorphic, and various weak solutions correspond to various weak states. In other words, in the semigroup of transformations generated by the system with delay and acting on the space of weak states, abbreviations are possible and, thus, this semigroup can be reduced to its group of partials.

T. B. Kopeikina has studied singular perturbed systems with aftereffect from the point of view of controllability and observability based on the Vasil'eva's method of boundary functions.

In conclusion, let us list other priority directions of the studies of Belarussian scientists on QCOT in systems, which for some reasons were not considered in detail in the present study: conditional and relative observability (R. Gabasov, F. M. Kirillova, etc.), controllability (A. I. Astrovskii, V. V. Ignatenko, B. Sh. Shklyar) and observability (V. V. Mulyarchik) in elementary classes of functions, calculation of the minimum number of inputs and outputs (V. M. Marchenko), splitability (I. K. Asmykovich), the problem of reconstruction (V. I. Yanovich), allied problems of control and observation (R. Gabasov, F. M. Kirillova, etc.), canonical representations of controllable systems (F. M. Kirillova, V. M. Marchenko, V. L. Merezha), some problems of identification and observability (A. I. Astrovskii, A. V. Metel'skii, S. A. Minyuk), nonstationary systems (R. Gabasov and F. M. Kirillova, A. I. Astrovskii, L. E. Zabello, T. B. Kopeikina, V. P. Kirlitsa, V. M. Marchenko, B. S. Mordukhovich, etc.), etc.

Thus, we have outlined main problems of QCOT in Belarus to the present moment.

4. TOTALLY REGULAR SYSTEMS WITH A DELAYED ARGUMENT. SOME UNSOLVED PROBLEMS

At present, systems (differential-algebraic, descriptor) unsolved with respect to a derivative are of increasing interest.

Let us consider an elementary differential-algebraic system with a delayed argument in the state and control

$$\frac{d}{dt}(Hx(t)) = Ax(t) + A_1x(t-h) + Bu(t) + Pu(t-h), \quad t \ge 0, \tag{12}$$

with the initial conditions,

$$Hx(+0) = Hx(0) = Hx_0, \quad A_1x(\tau) = A_1\varphi(\tau), \quad \tau \in [-h,0), \tag{13}$$

where $H, A, A_1 \in R^{n\times n}$, $B \in R^{n\times r}$, $P \in R^{n\times r}$, $x_0 \in R^n$, φ is an n-vector function piecewise-continuous on $[-h,0]$, and u is a piecewise-continuous control action (control), for $t = 0$ the right-hand derivative is considered in (12).

Definition 1. We will call system (11)

a) strictly regular if $\det H \ne 0$,

b) k-completely regular if $\lim\limits_{s\to+\infty}\left(\frac{1}{s}\det(sH - A - e^{-sh}A_1)\right) \ne 0$, where $k = \operatorname{rank} H$ (completely regular if $1 \le k < n$),

c) regular if $\det(sH - A - e^{-sh}A_1) \not\equiv 0$ for $s \in C$, where C is a field of complex numbers.

The basic problems of QCOT for strictly regular systems were analyzed in the previous sections.

It is easy to show that any completely regular system (12) can be written in an equivalent form of a hybrid system with a delayed argument

$$\begin{cases} \dot{x}_1(t) = A_{11}x_1(t) + A_{11}^1x_1(t-h) + A_{12}x_2(t) + A_{12}^1x_2(t-h) + B_1u(t) + B_1^1u(t-h), \\ x_2(t) = A_{21}x_1(t) + A_{21}^1x_1(t-h) + A_{22}^1x_2(t-h) + B_2u(t) + B_2^1u(t-h), t \ge 0, \end{cases} \tag{14}$$

where $x_1(t) \in \mathbf{R}^{n_1}$, $x_2(t) \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$; $u(t) \in \mathbf{R}^r$, $t \ge 0$.

As examples of hybrid systems, let us consider a classical linear control system with the output

$$\dot{x}(t) = Ax(t) + Bu(t),$$

$$y(t) = Cx(t) + Du(t)$$

(here $x_1(t) = x(t)$ and $x_2(t) = y(t)$) and the system

$$\frac{d}{dt}(x(t) - Dx(t-h)) = Ax(t) + A_1 x(t-h) + Bu(t)$$

with a delayed argument of neutral type (here $x_1(t) = x(t) - Dx(t-h)$, $x_2(t) = x(t)$). As we see, the considered system of neutral type is a special case of an elementary descriptor system (12).

Note that the inverse reduction of a hybrid system to a system of neutral type by, for example, differentiating the second equation in (14) is unacceptable by virtue of the first one since it significantly narrows down the set of solutions of system (14). In the general case, it is easy to see that (14) can be reduced to a system with an infinite number of delays and finite initial data, however, this does not simplify its analysis. Thus, an immediate analysis of hybrid systems with regard for their specificity seems to be expedient.

Let us consider an elementary controlled hybrid system with a delayed argument

$$\dot{x}(t) = A_{11}x(t) + A_{12}y(t) + B_1u(t), \quad t > 0, \tag{15}$$

$$y(t) = A_{21}x(t) + A_{22}^1 y(t-h) + B_2u(t), \quad t \geq 0,$$

with the initial conditions

$$x(+0) = x_0, \ A_{22}^1 y(\tau) = A_{22}^1 \psi(\tau), \tau \in [-h, 0). \tag{16}$$

It turns out (V. M. Marchenko, O. N. Poddubnaya) that a solution of system (15) corresponding to the piecewise-continuous control $u(\cdot)$ and initial data (16) with the piecewise-continuous vector function $\psi(\cdot)$ exists, is unique, and has an integral representation, which can be expressed, by analogy with [9], by the solutions of a conjugate system. This generalizes the well-known presentation of the solution [7, 9] by the Cauchy formula to systems (15), (16). Note that the equation of jumps and different boundary-value conditions for representation of $x(t)$ and $y(t)$ for $t \geq 0$ appear additionally in the conjugate system. The exponential estimate of the solutions holds (that allows application of Laplace transformation to system (15), (16)), and also representation of the solutions as a series in the solutions of constitutive equations (that generalizes representation (5) to systems (15), (16)).

In conclusion, let us formulate some unsolved problems of QCOT in systems with aftereffect.

Problem 1. To obtain a criterion of (pointwise) controllability and observability of completely regular (hybrid) systems with aftereffect which would relate this controllability and observability to the Kalman controllability and observability of special parametric systems without delay.

Problem 2. To obtain a (parametric) criterion of total controllability of quire regular systems.

Problem 3. To analyze controllability of completely regular systems with aftereffect of the feedback type: stabilizability, modal control, reconstruction, etc.

Problem 4. To study existence, uniqueness, exponential estimate, and representation of solutions of an elementary descriptor system, in particular, regular systems as a nearest generalization of completely regular systems.

It is well known [3] that transformation over a ring of polynomials with respect to a shift operator does not withdraw the considered system from the class of strictly regular systems with many commensurable delays while transformation over a field of partials of the above-mentioned ring leads, in the general case, to a (more general) transformed system of neutral type. In this context, note that the class of descriptor systems with many delays is invariant with respect to the above-mentioned transformations.

Problem 5. To obtain canonical representations of various classes of systems of the form (14) subject to various groups of transformations [2, 5, 32]. To study main problems of QCOT for such systems.

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