Medical Image Analysis

CS 778 / 578

Computer Science and Electrical Engineering Dept.
West Virginia University

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Outline

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Using sliding windows for image processing

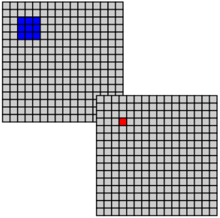


Figure: Pixel [i,j] in the output image (shown in red) depends on the pixels in the neighborhood of [i,j] in the input image (shown in blue).

(i-1, j+1)	(i, j+1)	(i+1, j+1)
(i-1, j)	(i, j)	(i+1, j)
(i-1, j-1)	(i, j-1)	(i+1, j-1)

Coordinates in the 3×3 neighborhood of pixel (i,j).

Image smoothing

We can compute several things within each window:

- Local averaging : mean
- Weighted local average : weight the center pixel higher
- Nonlinear filtering : median

- We can look at larger neighborhoods : 5×5, 7×7 and larger.
- Why are these odd dimensions?
- In 3D : $3 \times 3 \times 3$ neighborhoods and larger.

Local averaging

We associate weights, w_{km} with each voxel in the sliding window:

$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

The new intensity, g(i,j), computed from from the original image, f, and the weights, w

$$g(i,j) = \sum_{m=-1}^{1} \sum_{k=-1}^{1} w_{km} f(i+k, j+m)$$

is the mean of intensity values within the sliding window.

When applied to the whole image we write $g = w \otimes f$
(This is also referred to as 'convolution')

Local averaging results

- Edges are destroyed.
- Smoothing is a low-pass filtering operation.
- High frequency components are attenuated, low frequency components of the image are preserved.

It can be shown that if image f is corrupted by additive Gaussian noise of variance σ_f^2 then the variance of the noise on image g is

$$\sigma_g^2 = \sum_{m=-1}^{1} \sum_{k=-1}^{1} w_{km}^2 \sigma_f^2$$

Fourier transform

Transform from time/space domain to frequency domain

Denoted $\mathcal{F}[f] = \hat{f}$

Some calculations are simpler in frequency domain:

Function	Fourier transform unitary, ordinary frequency
$f(x)$	$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$
$a \cdot f(x) + b \cdot g(x)$	$a \cdot \hat{f}(\xi) + b \cdot \hat{g}(\xi)$
$f(x-a)$	$e^{-2\pi i a \xi} \hat{f}(\xi)$
$e^{2\pi i a x} f(x)$	$\hat{f}(\xi - a)$
$f(ax)$	$\frac{1}{\lvert a \rvert} \hat{f}\left(\frac{\xi}{a}\right)$
$\hat{f}(x)$	$f(-\xi)$
$\frac{d^n f(x)}{dx^n}$	$(2\pi i \xi)^n \hat{f}(\xi)$

Fourier Transform - Convolution Theorem

The continuous definition of convolution

$$f * g = \int_{-\infty}^{\infty} f(\tau) g(x - \tau)\, d\tau \tag{1}$$

$$\mathcal{F}[f * g] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) g(x - \tau)\, d\tau \right] e^{-2\pi i k x}\, dx \tag{2}$$

Using the fact that $e^x = e^\tau e^{x-\tau}$

$$\mathcal{F}[f * g] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [e^{-2\pi i k \tau} f(\tau)][e^{-2\pi i k (x-\tau)} g(x - \tau)]\, dx\, d\tau \tag{3}$$

Substituting $y = x - \tau$, and $dy = dx$

$$\mathcal{F}[f * g] = \int_{-\infty}^{\infty} e^{-2\pi i k \tau} f(\tau)\, d\tau \int_{-\infty}^{\infty} e^{-2\pi i k y} g(y)\, dy \tag{4}$$

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g] \tag{5}$$

- Convolution can be implemented as multiplication in the frequency domain.
- Gaussian convolution is Gaussian multiplication in frequency domain.
- Gaussian convolution is low-pass filtering.

Image Gradient

$$\nabla I(x,y) = \begin{bmatrix} \frac{\partial I}{\partial x} \\ \\ \frac{\partial I}{\partial y} \end{bmatrix} \tag{6}$$

Vector points in the direction of fastest increase in $I(x,y)$

- Edges are discontinuities in intensity
- $||\nabla I||$ is the edge "strength" (sharpness of the edge)
- ∇I is perpendicular to the edge

Edge detection using the gradient

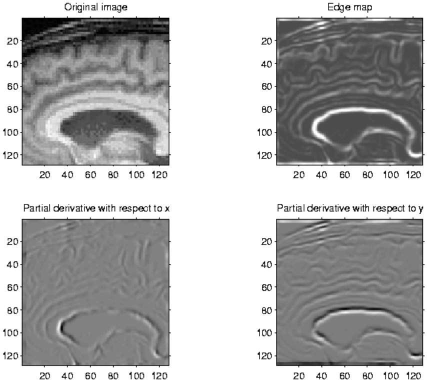


Image from "Visualizing calculus: The use of the gradient in image processing"

(http://amath.colorado.edu)

Divergence

For the vector-valued function $\mathbf{V}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} v_1(x, y) \\ v_2(x, y) \end{bmatrix}$ the divergence is defined as

$$\text{div}\, \mathbf{V}(\mathbf{x}, \mathbf{y}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \tag{7}$$

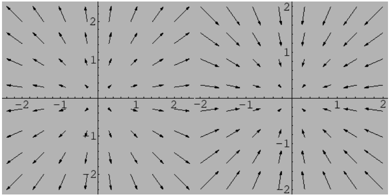
More notation:

$\text{div}(\nabla I) = \nabla \cdot \nabla I = \nabla^2 I$ (the Laplacian)

Divergence of a vector field

$$\operatorname{div} \mathbf{V}(\mathbf{x}, \mathbf{y}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \tag{8}$$

Physical interpretation : sources or sinks a velocity field.



div $V > 0$ (left), div $V < 0$ (right)
Image from "The idea of divergence and curl"

(http://www.math.umn.edu)

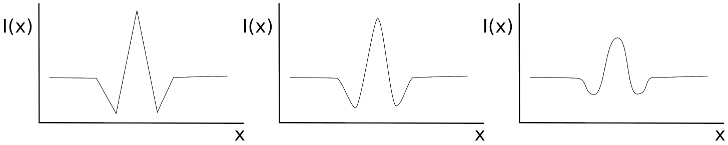
Outline

The Heat Equation / Diffusion Equation

Let image = $I(x, y, z, t)$.

$$\frac{\partial I}{\partial t} = \text{div}(\nabla I) \tag{9}$$

- The equation describes the way physical systems achieve equilibrium.
- When describing heat transfer, $I(x)$ is temperature.
- When describing diffusion, $I(x)$ is molecular concentration.



Solving the heat equation

In one dimension:

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} \tag{10}$$

Take the Fourier transform of both sides

$$\mathcal{F}\left[\frac{\partial I}{\partial t}\right] = \mathcal{F}\left[\frac{\partial^2 I}{\partial x^2}\right] \tag{11}$$

Let $\mathcal{F}[I(x,t)] = U(\omega, t)$ and

Recall that

if $f(x) \leftrightarrow F(\omega)$,
then $f'(x) \leftrightarrow i\omega F(\omega)$, and
$f''(x) \leftrightarrow -\omega^2 F(\omega)$

The Heat Equation / Diffusion Equation

$$\frac{\partial U(\omega, t)}{\partial t} = -\omega^2 U(\omega, t) \tag{12}$$

with initial conditions

$$U(\omega, 0) = U_0(\omega) \tag{13}$$

You can verify that

$$U(\omega, t) = U_0(\omega) e^{-\omega^2 t} \tag{14}$$

Is a solution to the problem.

The Heat Equation / Diffusion Equation

Taking the inverse Fourier transform

$$\mathcal{F}^{-1}[U(\omega,t)] = \mathcal{F}^{-1}[U_0(\omega)e^{-\omega^2 t}] \tag{15}$$

we see that

$$I(x,t) = I_0(x) * e^{-\frac{x^2}{2\sigma_t^2}} \tag{16}$$

where $\sigma_t^2 = 2t$

This is Gaussian convolution

The Heat Equation / Diffusion Equation

- Evolving $I(x)$ according to the heat equation is equivalent to convolving with a Gaussian kernel.
- Longer evolution corresponds to convolution with Gaussians of higher variance.
- This gives us firm footing for studying PDEs in the context of image processing.

Next Class

Variational calculus.
Numerical methods for solving the heat equation.