Medical Image Analysis

CS 778 / 578

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January 24, 2011

Outline

1. Discretizing the heat / diffusion equation
2. Matrix forms of the discretized heat equation
3. Scale space image representation

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1. Discretizing the heat / diffusion equation
   - Numerical differentiation
   - First derivative approximations
   - Second derivative approximations
2. Matrix forms of the discretized heat equation
3. Scale space image representation

Introduction

The names 'heat equation' and 'diffusion equation' are used interchangeably: the same equation describes both phenomena.

$$\begin{aligned}
\text{In 1D} \quad \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \\
\text{In 2D} \quad \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
\text{In 3D} \quad \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}
\end{aligned}$$

$$\text{In general} \quad \frac{\partial u}{\partial t} = \operatorname{div}(\nabla u)$$

- Heat : u is temperature
- Diffusion : u is concentration
- Images : u is image intensity

Introduction

- We know the heat equation can be solved analytically.
  - $I(x,t) = I_0(x) * e^{-\frac{x^2}{2\sigma_t^2}}$
- In the future we will consider nonlinear variants of the diffusion equation for which no simple analytical solution exists.
- In these cases we approximate a solution numerically.

Introduction

Problem

Estimate derivatives of some unknown smooth function, $f(x)$, given only samples $\{f(x_i)\}$.

Motivation

Find approximate solutions to PDEs governing evolution of $f(x)$.

Taylor Series

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \ldots \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0)$$

Taylor series expansion is the basis for many numerical methods. For example : numerical differentiation, Newton's method...

Forward Difference Equation

Expand $f(x)$ in a Taylor series about x_0.

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0)$$

Then evaluate at $x = x_0 + h$.

$$f(x_0 + h) \approx f(x_0) + hf'(x_0)$$

First order forward difference:

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

Backward Difference Equations

Replace h with $-h$ in the previous derivation

$$f(x_0 - h) \approx f(x_0) - hf'(x_0)$$

First order backward difference:

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h}$$

Centered Difference Equation

Subtract the first order expansions:

$$f(x_0 + h) \approx f(x_0) + hf'(x_0)$$

$-$

$$f(x_0 - h) \approx f(x_0) - hf'(x_0)$$

$=$

$$f(x_0 + h) - f(x_0 - h) \approx 2hf'(x_0)$$

Divide by $2h$ to get the centered difference:

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

Second Centered Difference Equation

Second order expansion evaluated at $x = x_0 + h$

$$f(x_0 + h) \approx f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0)$$

Second order expansion evaluated at $x = x_0 - h$

$$f(x_0 - h) \approx f(x_0) - hf'(x_0) + \frac{h^2}{2} f''(x_0)$$

Second Centered Difference Equation

We can approximate the **second derivative** by adding two second order expansions:

$$f(x_0 + h) \approx f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0)$$

$+$

$$f(x_0 - h) \approx f(x_0) - hf'(x_0) + \frac{h^2}{2} f''(x_0)$$

$=$

$$f(x_0 + h) + f(x_0 - h) \approx 2f(x_0) + h^2 f''(x_0)$$

Second Centered Difference Equation

Rearrange

$$f(x_0 + h) + f(x_0 - h) \approx 2f(x_0) + h^2 f''(x_0)$$

to get the (second order) second centered difference

$$f''(x_0) \approx \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}$$

Error analysis

The Taylor remainder : estimate of how well the Taylor series approximates a function

$$f(x_0+h) = f(x_0) + hf'(x_0) + O(h^2)$$

First order forward difference:

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} + \frac{O(h^2)}{h}$$

So the error is $O(h)$. Same analysis holds for the backward difference approximation.

Error analysis : central difference

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + O(h^3)$$

$-$

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) + O(h^3)$$

$=$

$$f(x_0 + h) - f(x_0 - h) = 2hf'(x_0) + O(h^3)$$

Divide by $2h$ to get the centered difference:

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} + O(h^2)$$

Error analysis : second central difference

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + O(h^4)$$

$+$

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + O(h^4)$$

$=$

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + h^2 f''(x_0) + O(h^4)$$

So the (second order) second centered difference is

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} + O(h^2)$$

Outline

1. Discretizing the heat / diffusion equation
2. Matrix forms of the discretized heat equation
    - Forward difference method
    - Backward difference method
3. Scale space image representation

Explicit Method

Recall the heat equation

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}$$

Use forward difference in time and second central difference in space:

$$\frac{I_{x,y}^{t+\delta} - I_{x,y}^{t}}{\delta} = (I_{x+1,y}^{t} - 2I_{x,y}^{t} + I_{x-1,y}^{t}) + (I_{x,y+1}^{t} - 2I_{x,y}^{t} + I_{x,y-1}^{t})$$

- Superscripts : time
- Subscripts : position

Forming the linear system of equations

$$I_{x,y}^{t+\delta} = I_{x,y}^{t} + \delta(I_{x+1,y}^{t} - 4I_{x,y}^{t} + I_{x-1,y}^{t} + I_{x,y+1}^{t} + I_{x,y-1}^{t})$$

- We want to simultaneously solve for $I_{x,y}^{t+\delta}$ at all x, y.
- The image can be stretched into a vector, **w**, by stacking the columns of the image on top of each other.
  - Matlab : `reshape` command

$$I = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix} \rightarrow \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 16 \end{bmatrix}$$

The evolution equation will be $\mathbf{w}^{t+\delta} = \mathbf{A}\mathbf{w}^{\mathbf{t}}$

Reshaping image I into array $\mathbf{w}$

$$I = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix} \rightarrow \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ 16 \end{bmatrix}$$

allows us to rewrite the equations

$$I_{x,y}^{t+\delta} = I_{x,y}^{t} + \delta(I_{x+1,y}^{t} - 4I_{x,y}^{t} + I_{x-1,y}^{t} + I_{x,y+1}^{t} + I_{x,y-1}^{t})$$

in terms of $\mathbf{w}$.
For example,

$$w_{10}^{t+\delta} = w_{10}^{t} + \delta(w_{14}^{t} - 4w_{10}^{t} + w_{6}^{t} + w_{11}^{t} + w_{9}^{t})$$

this can be rewritten as a row vector times a column vector

$$w_{10}^{t+\delta} = \begin{bmatrix} \dots & \delta & \dots & \delta & 1-4\delta & \delta & \dots & \delta & \dots \end{bmatrix} \mathbf{w}$$

Forming the linear system of equations

In general we have

$$w_i^{t+\delta} = w_i^t + \delta(w_{i+1}^t - 4w_i^t + w_{i-1}^t + w_{i+n}^t + w_{i-n}^t)$$

- Collect the coefficients of $\mathbf{w}$ into a matrix, $\mathbf{A}$.
- If the image, $I(x, y)$ is size $n \times n$, the vector, $\mathbf{w}$, is $n^2 \times 1$,
- the matrix of coefficients, $\mathbf{A}$ is $n^2 \times n^2$.
- Most elements of $\mathbf{A}$ are 0 (i.e. $\mathbf{A}$ is sparse)

Linear system of equations

$$w_i^{t+\delta} = w_i^t + \delta(w_{i+1}^t - 4w_i^t + w_{i-1}^t + w_{i+n}^t + w_{i-n}^t)$$

The matrix of coefficients

$$\mathbf{A} = \begin{pmatrix} (1-4\delta) & \delta & 0 & \dots & \delta & \dots & \dots & 0 & 0 \\ \delta & (1-4\delta) & \delta & 0 & \dots & \delta & \dots & \dots & 0 \\ 0 & \delta & (1-4\delta) & \delta & 0 & \dots & \delta & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \ddots & \end{pmatrix}$$

Properties of **A**:

- Symmetric,
- Sparse:
  - 5 nonzero diagonals for a 2D image
  - 7 nonzero diagonals for a 3D image
  - Matlab : Use `sparse, spdiags`

Explicit or forward solution

Each iteration is a matrix multiplication.

$$\mathbf{w}^{t+\delta} = \mathbf{A}\mathbf{w}^{t}$$

Convergence Criterion

Steady-state is reached when $\mathbf{w}^{t+\delta} \approx \mathbf{w}^{t}$.
Check $||\mathbf{w}^{t+\delta} - \mathbf{w}^{t}|| < \epsilon$

Problem

- For large δ we may overshoot the solution.
- The iteration will oscillate, and never converge.
- Stability is only guaranteed for small δ, and then convergence is slow.

Implicit Method

Recall the heat equation

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}$$

Use **backward difference** in time and second central difference in space:

$$\frac{I^t_{x,y} - I^{t-\delta}_{x,y}}{\delta} = (I^t_{x+1,y} - 2I^t_{x,y} + I^t_{x-1,y}) + (I^t_{x,y+1} - 2I^t_{x,y} + I^t_{x,y-1})$$

Linear system for backward difference method

The generic difference equation

$$I_{x,y}^{t-\delta} = I_{x,y}^{t} - \delta(I_{x+1,y}^{t} - 4I_{x,y}^{t} + I_{x-1,y}^{t} + I_{x,y+1}^{t} + I_{x,y-1}^{t})$$

has the vector form

$$w_i^{t-\delta} = w_i^t - \delta(w_{i+1}^t - 4w_i^t + w_{i-1}^t + w_{i+n}^t + w_{i-n}^t)$$

Linear system of equations

$$\mathbf{w}^{t-\delta} = \mathbf{B}\mathbf{w}^{t}$$

The matrix of coefficients

$$\mathbf{B} = \begin{pmatrix} (1+4\delta) & -\delta & 0 & \dots & -\delta & \dots & \dots & 0 & 0 \\ -\delta & (1+4\delta) & -\delta & 0 & \dots & -\delta & \dots & \dots & 0 \\ 0 & -\delta & (1+4\delta) & -\delta & 0 & \dots & -\delta & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \ddots & \end{pmatrix}$$

Properties of **B**:

- Symmetric,
- Sparse:
  - 5 nonzero diagonals for a 2D image
  - 7 nonzero diagonals for a 3D image

Implicit or backward solution

Each iteration requires solution of a linear system (inversion or factorization).

$$\mathbf{B}\mathbf{w}^t = \mathbf{w}^{t-\delta}$$

This method is stable, however setting δ too large will result in slow convergence.

Implementation

- Don't simply invert B and compute $\mathbf{w}^t = \mathbf{B}^{-1}\mathbf{w}^{t-\delta}$
- Factorize (Cholesky or LU) and solve
- In Matlab : `w = B \w`

Matrix stability analysis

If we have some small error, $\mathbf{e}^0$ in the initial condition $\mathbf{w}^0$

$$\mathbf{w}^1 = \mathbf{A}(\mathbf{w}^0 + \mathbf{e}^0) = \mathbf{A}\mathbf{w}^0 + \mathbf{A}\mathbf{e}^0$$

and at the next iteration we have

$$\mathbf{w}^2 = \mathbf{A}(\mathbf{A}\mathbf{w}^0 + \mathbf{A}\mathbf{e}^0) = \mathbf{A}^2\mathbf{w}^0 + \mathbf{A}^2\mathbf{e}^0.$$

In general, at iteration n, we have

$$\mathbf{w}^n = \mathbf{A}^n\mathbf{w}^0 + \mathbf{A}^n\mathbf{e}^0.$$

Whether $||\mathbf{A}^n\mathbf{e}^0|| \leq ||\mathbf{e}^0||$ depends on the **condition number** of matrix $\mathbf{A}$.

Condition number(A) ≈ 1 is well-conditioned.

Matrix stability analysis

Computing the condition number is difficult, but...

As a general rule:

Strictly diagonally dominant matrices are well-conditioned.

Definition : A matrix, A, is **strictly diagonally dominant** if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all rows i.

Stability of the forward difference equation

A typical row of matrix **A**:

$$\mathbf{A}_i = \begin{pmatrix} 0 & \dots & \delta & 0 & \dots & \delta & (1-4\delta) & \delta & 0 & \dots & \delta & 0 & \dots \end{pmatrix}$$

For what values of δ is the matrix diagonally dominant? Values of δ which satisfy the inequality

$$|1-4\delta| > 4|\delta|$$

Recall : How to handle inequalities with absolute values

Consider the inequality

$$|x| > 5$$

There are two intervals which satisfy it : a positive interval

$$x > 5$$

and a negative interval

$$x < -5$$

Stability of the forward difference equation

What values of δ satisfy

$$|1 - 4\delta| > 4|\delta|?$$

First, rearrange to get the absolute value on left-hand side

$$\frac{|1 - 4\delta|}{4|\delta|} > 1$$

then simplify the two intervals :

$$\frac{1 - 4\delta}{4\delta} > 1$$

and

$$\frac{1 - 4\delta}{4\delta} < -1$$

Stability of the forward difference equation

Simplifying the first interval:

$$\frac{1 - 4\delta}{4\delta} > 1$$

$$1 - 4\delta > 4\delta$$

$$1 > 8\delta$$

This is satisfied by $\delta < \frac{1}{8}$

Stability of the forward difference equation

Simplifying the second interval:

$$\frac{1-4\delta}{4\delta} < -1$$

$$1 - 4\delta < -4\delta$$

$$1 < 0$$

This is impossible to satisfy, so the second interval is empty.
So the inequality

$$|1 - 4\delta| > 4|\delta|$$

is satisfied (and the forward difference method is stable) only for $\delta < \frac{1}{8}$.

Stability of the backward difference equation

A typical row of matrix **B**:

$$\mathbf{B}_i = \begin{pmatrix} 0 & \dots & -\delta & 0 & \dots & -\delta & (1+4\delta) & -\delta & 0 & \dots & -\delta & 0 & \dots \end{pmatrix}$$

For what values of δ is the matrix diagonally dominant?

$$|1 + 4\delta| > 4|\delta|$$

$$\frac{1+4\delta}{4\delta} > 1 \rightarrow 1 + 4\delta > 4\delta \rightarrow 1 > 0$$

or

$$\frac{1+4\delta}{4\delta} < -1$$

The backward difference method is stable for all δ. (Unconditionally stable)

Matrix stability analysis

For more details about matrix condition number and spectral radius, see *Golub and Van Loan*, "**Matrix Computations**" or another numerical linear algebra text.

In Matlab use `cond, condest`

Implicit or backward solution

Each iteration requires solution of a linear system (inversion or factorization).

$$\mathbf{B}\mathbf{w}^t = \mathbf{w}^{t-\delta}$$

This method is unconditionally stable, however setting δ too large will result in slow convergence.

Problem

Error is of order $O(\delta)$.

Mixed Explicit/Implicit Method

We can get order $O(\delta^2)$ error by averaging the two difference equations. Note that the second order terms in the Taylor series cancel, just like they did when we computed central differences.

$$\frac{1}{2}\mathbf{w}^{t+\delta} = \frac{1}{2}\mathbf{A}\mathbf{w}^{t}$$

+

$$\frac{1}{2}\mathbf{B}\mathbf{w}^{t+\delta} = \frac{1}{2}\mathbf{w}^{t}$$

=

$$(\mathbf{B}+\mathbf{I})\mathbf{w}^{t+\delta} = (\mathbf{A}+\mathbf{I})\mathbf{w}^{t}$$

Solutions to the linear system

$$(\mathbf{B} + \mathbf{I})\mathbf{w}^{t+\delta} = (\mathbf{A} + \mathbf{I})\mathbf{w}^{t}$$

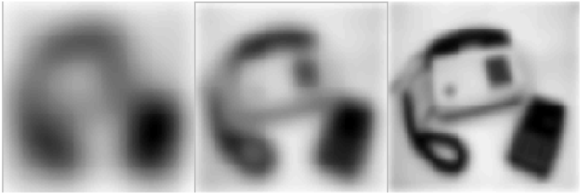
- Don't try to invert the matrix $\mathbf{B} + \mathbf{I}$.
- Instead use Gaussian elimination, LU decomposition.

Outline

1. Discretizing the heat / diffusion equation
2. Matrix forms of the discretized heat equation
3. Scale space image representation
   - The concept
   - A requirement
   - Next Class

What is scale space?

We may be interested in image features which exist at different scales in the image, so we want to represent an image over a continuum of scales (coarse to fine).



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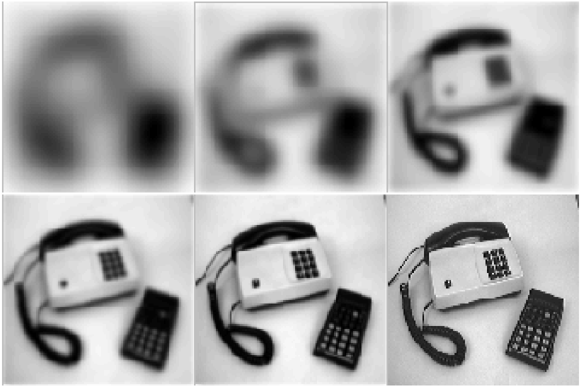


Scale-space requirement

As we progress from fine to coarse through scale space, we should not create new details.

- Images generated by isotropic diffusion satisfy this requirement.
- Equivalent to convolution / low-pass filtering
- However, edges at the coarse scales are blurred.
- We must track features up to the finest scale to get their true locations.

New scale space representation

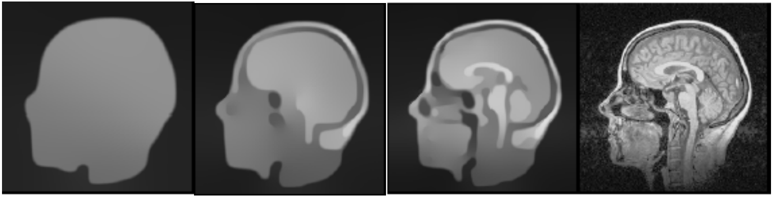
Perona and Malik suggest a new technique for generating the scale space of images which preserves edges in "*Scale-Space and Edge Detection Using Anisotropic Diffusion*". They propose the use of

- **Inhomogeneous diffusion**: rate of diffusion varies spatially.

Weickert, in "*A Review of Nonlinear Diffusion Filtering.*", proposes

- **Anisotropic diffusion**: rate of diffusion at a point varies with direction.

to generate the scale space of images and perform denoising.



Start reading the Perona/Malik paper

- The physical process of diffusion.
- Discuss the Perona/Malik paper.