Medical Image Analysis

CS 778 / 578

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West Virginia University

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Outline

Outline

Stability and Convergence

- Stability: noise (from initial conditions, round-off error) is not amplified.
- Convergence: numerical scheme approaches solution of the PDE as $t \to \infty$

Convergence of the explicit 1D heat equation

The 1D heat equation, $I_t = I_{xx}$, has solution $I(x,t) = e^{-t}\cos(x)$.
This corresponds to the problem with initial condition $I(x,0) = \cos(x)$.

Discretize only in time (forward)

Observe that $I_{xx}(x,t) = -e^{-t}\cos(x) = -I(x,t)$

$$\frac{I^{t+\delta} - I^t}{\delta} = -I^t$$

$$I^{t+\delta} = I^t - \delta I^t$$

Convergence criterion : ratio test

The sequence I^t is convergent if

$$\lim_{t\to\infty}\left|\frac{I^{t+\delta}}{I^t}\right| < 1$$

The explicit equation we formed earlier

$$I^{t+\delta} = I^t - \delta I^t$$

has convergence criterion

$$\left|\frac{I^{t+\delta}}{I^t}\right| = |1-\delta| < 1$$

This is satisfied for $0 < \delta < 2$. (Only conditionally convergent.)

Convergence of the implicit 1D heat equation

Discretize only in time (backward)

$$\frac{I^{t+\delta} - I^t}{\delta} = -I^{t+\delta}$$

$$I^{t+\delta} = I^t - \delta I^{t+\delta}$$

The implicit equation has convergence criterion

$$\left| \frac{I^{t+\delta}}{I^t} \right| = \left| \frac{1}{1+\delta} \right| < 1$$

This is satisfied for $\delta > 0$.
Backward heat equation does not converge in either case.

Convergence

In general, it can be shown that

- Explicit methods are conditionally convergent.
- Implicit methods are unconditionally convergent.

Outline

Flux definition

Flux : Rate of movement of *something* per unit area.

What is moving?

- Diffusion: molecules
- Heat: energy

For diffusion the units of flux are $\frac{mol}{m^2 s}$

Flux

Isotropic Diffusion: $j = -d\nabla u$

Fick's First Law : Molecules diffuse from high concentration to low concentration.

- d is a scalar diffusivity constant
- Flux is parallel to concentration gradient, but in opposite direction.

Heat: $q'' = -k\nabla T$

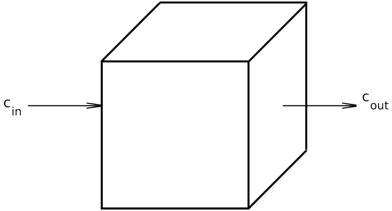
Heat flows from high temperature to low temperature.

Anisotropic Diffusion: $j = -\mathbf{D}\nabla u$

Concentration gradient causes a flux which is transformed by the matrix **D**.

Discrete example

Let c be the number of molecules.

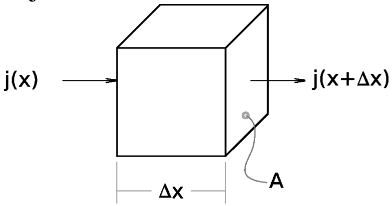


$$c_{in} - c_{out} = \Delta c_{stored}$$

Matter/energy is not created or destroyed.

Discrete example

Let j be molecular flux over time Δt



$$(j(x) - j(x+\Delta x))A = \frac{\Delta c_{stored}}{\Delta t}$$

Matter/energy is not created or destroyed.

Extend this example to 2D , and 3D...

Fick's Second Law

Conservation of Mass

$$\frac{\partial u}{\partial t} = -\operatorname{div} j$$

with Fick's First Law ($j = -d\nabla u$) yields the diffusion equation

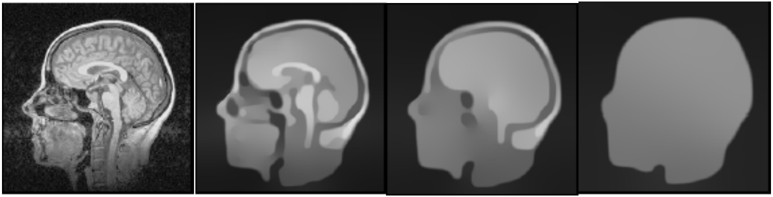
$$\frac{\partial u}{\partial t} = \operatorname{div}(d\nabla u)$$

- Perona-Malik idea : Make d inhomogeneous (d(x,y))
  - Slow down / speed up diffusion as needed*
- Weikert idea : Make d anisotropic (D(x,y))
  - Direct flux as needed

Outline

Scale-space

The need for multiscale image representations: Details in images should only exist over certain ranges of scale.



Scale-space

Definition: a family of images, $I(x, y, t)$, where

- The scale-space parameter is t.
- $I(x, y, 0)$ is the original image.
- Increasing t corresponds to coarser resolutions.

$I(x, y, t)$ can be generated by convolving with wider Gaussian kernels as t increases, or equivalently, by solving the heat equation.

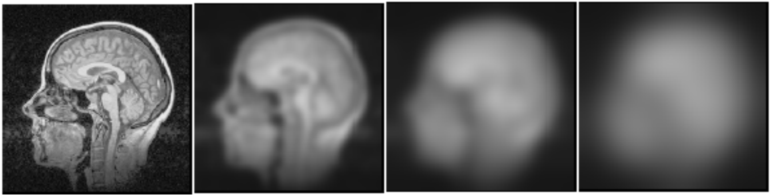
Earlier Scale-space properties

- Causality: coarse details are ”caused” by fine details.
- New details should not arise in coarse scale images.
- Smoothing should be homogeneous and isotropic.

This paper will challenge the last property, and propose a more useful scale-space definition.
The new scale-space will be shown to obey the causality property.

Lost Edge Information

- Edges may disappear.
- Edge location is not preserved across the scale space.
- Region boundaries are blurred.



Gaussian blurring is a local averaging operation. It does not respect natural boundaries.

Linear Scale Space

Def: Scale spaces generated by a linear filtering operation.

- Nonlinear filters, such as the median filter, can be used to generate **nonlinear** scale-spaces.
- Many nonlinear filters violate one of the scale-space conditions.

New Criteria

- Causality.
- Immediate localization : edge locations remain fixed.
- Piecewise Smoothing : permit discontinuities at boundaries.

At all scales the image will consist of smooth **regions** separated by **boundaries** (edges).

Diffusion equation

$$\frac{\partial I}{\partial t} = \operatorname{div}(c(x, y, t)\nabla I)$$

The diffusion coefficient, $c(x, y, t)$ controls the degree of smoothing at each point in I.

The basic idea:

Setting $c(x, y, t) = 0$ at region boundaries, and $c(x, y, t) = 1$ at region interior will encourage **intraregion** smoothing, and discourage **interregion** smoothing.

Conduction coefficient

What properties would we like $c(x, y, t)$ to have?

- c = 1 at interior of a region.
- c = 0 at boundary of a region.
- c should be nonnegative everywhere.

Since $c(x, y, t)$ depends on edge information, we need an edge descriptor, $E(x, y, t)$, to compute c.

Notation

When written as a function of the edge descriptor, the authors use the symbol g() for conduction coefficient.

The function $g(||\nabla I||)$

Perona and Malik suggest two possible functions:

$$g(||\nabla I||) = e^{-(\frac{||\nabla I||}{K})^2}$$

$$g(||\nabla I||) = \frac{1}{1 + (\frac{||\nabla I||}{K})^{1+\alpha}} \quad (\alpha > 0)$$

Effect of varying K on $g(||\nabla I||)$

$$g(||\nabla I||) = \frac{1}{1 + (\frac{||\nabla I||}{K})^{1+\alpha}} \quad (\alpha > 0)$$

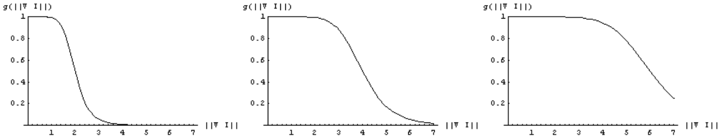


Figure: K = 2, 4, 6

Effect of varying α on $g(||\nabla I||)$

$$g(||\nabla I||) = \frac{1}{1 + (\frac{||\nabla I||}{K})^{1+\alpha}} \quad (\alpha > 0)$$

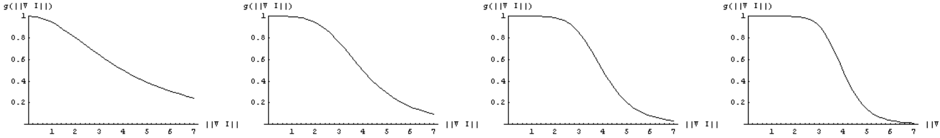


Figure: α = 1, 3, 5, 7, 9

Effect of varying K and α on $c(x, y)$

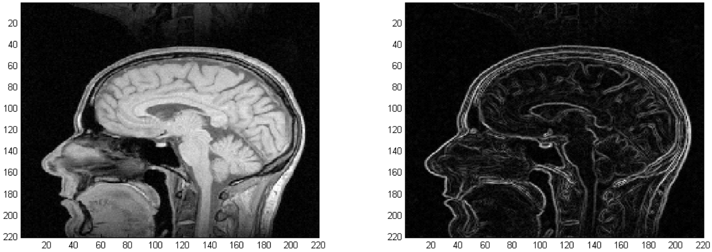


Figure: I and $||\nabla I||$.

Effect of varying K on $c(x, y)$

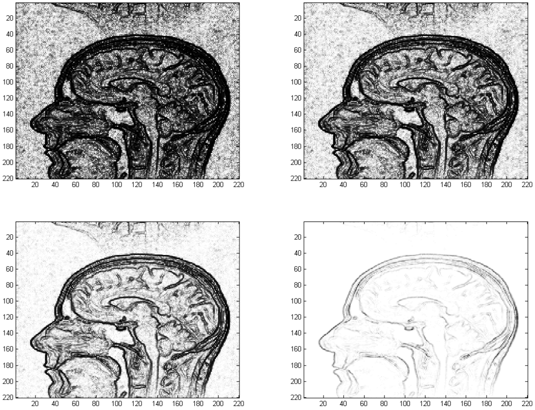


Figure: $K = 3, 5, 10, 100.$

As K increases, more edges will get smoothed out.

Effect of varying α on $c()$

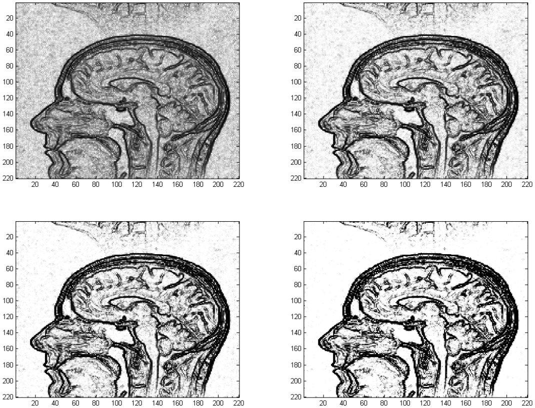
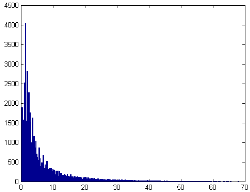


Figure: $\alpha = 1, 2, 3, 5.$

As α increases, the cutoff gets sharper.

Set K every iteration

Compute a histogram, f_i, of $||\nabla I||$



Find K such that 90% of the pixels have gradient magnitude $< K$.
(If $\sum_{i=1}^{b} f_i \geq 0.9n^2$ then bin b corresponds to gradient magnitude K).

Edge Enhancement

Inhomogeneous diffusion may actually enhance edges, for a certain choice of $c(x, y, t)$.

1D example:

Let $s(x) = \frac{\partial I}{\partial x}$, and $\phi(s) = g(I_x)I_x = g(s)s$.

The 1D inhomogeneous heat equation becomes

$$
\begin{aligned}
I_t = \frac{\partial}{\partial x}(g(I_x)I_x) &= \frac{\partial}{\partial x}\phi(s(x)) \\
\text{by chain rule} &= \frac{\partial \phi}{\partial s}\frac{\partial s}{\partial x} \\
I_t &= \phi'(s(x))I_{xx}
\end{aligned}
$$

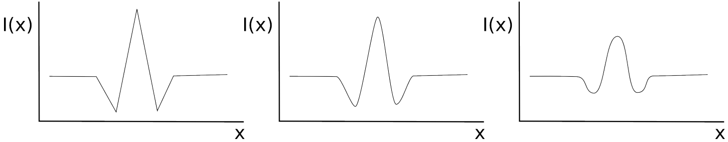
With a few clever substitutions you can identify the conditions for which $\frac{\partial}{\partial t}(I_x) > 0$. (See appendix of these notes.)

Maximum Principle

- The maximum and minimum intensities in the scale-space image $I(x, y, t)$ occur at $t = 0$ (the finest scale image).
- Since new maxima and minima correspond to new image features, the causality requirement of scale-space can satisfied if the evolution equation obeys the maximum principle.
- We will make some less rigorous observations concerning causality...

Maximum Principle

For the 1D heat equation : $I_t = I_{xx}$.



- Solving the heat equation is equivalent to convolution.
- Convolution is a local averaging operation.
- Averaging is bounded by the values being averaged.

Maximum Principle

For the Perona-Malik equation

$$\frac{\partial I}{\partial t} = c(x, y, t)\nabla^2 I + \nabla c \cdot \nabla I$$

Note that at local minima $\nabla I = \mathbf{0}$ and we are evolving by the original heat equation.

It can be shown that this general class of PDEs obeys the maximum principle.

See the 2009 class notes for a discussion of the maximum principle for the discretized equations.

Diffusion equation

By the chain rule:

$$\begin{aligned}
\frac{\partial I}{\partial t} &= \operatorname{div}\begin{pmatrix} c(x,y,t)\frac{\partial I}{\partial x} \\ c(x,y,t)\frac{\partial I}{\partial y} \end{pmatrix} \\
&= \frac{\partial c}{\partial x}\frac{\partial I}{\partial x} + c(x,y,t)\frac{\partial^2 I}{\partial x^2} + \frac{\partial c}{\partial y}\frac{\partial I}{\partial y} + c(x,y,t)\frac{\partial^2 I}{\partial y^2} \\
&= c(x,y,t)\nabla^2 I + \nabla c \cdot \nabla I
\end{aligned}$$

Notation

The paper uses the symbol Δ to represent the Laplacian.
$\Delta I = \nabla^2 I = \operatorname{div}(\nabla I)$

Explicit Formulation

$$\frac{\partial I}{\partial t} = c(x,y,t)\nabla^2 I + \nabla c \cdot \nabla I$$

Using centered differences for the Laplacian and gradients:

$$\begin{aligned}
\frac{I^{t+1}_{x,y} - I^{t}_{x,y}}{\lambda} &= c_{x,y}(I^t_{x-1,y} + I^t_{x+1,y} + I^t_{x,y-1} + I^t_{x,y+1} - 4I^t_{x,y}) \\
&+ (\frac{c_{x+1,y} - c_{x-1,y}}{2})(\frac{I_{x+1,y} - I_{x-1,y}}{2}) \\
&+ (\frac{c_{x,y+1} - c_{x,y-1}}{2})(\frac{I_{x,y+1} - I_{x,y-1}}{2})
\end{aligned}$$

Explicit Formulation

$$
\begin{aligned}
\frac{I_{x,y}^{t+1} - I_{x,y}^{t}}{\lambda} &= c_{x,y}(I_{x-1,y}^{t} + I_{x+1,y}^{t} + I_{x,y-1}^{t} + I_{x,y+1}^{t} - 4I_{x,y}^{t}) \\
&+ (\frac{c_{x+1,y} - c_{x-1,y}}{2})(\frac{I_{x+1,y} - I_{x-1,y}}{2}) \\
&+ (\frac{c_{x,y+1} - c_{x,y-1}}{2})(\frac{I_{x,y+1} - I_{x,y-1}}{2})
\end{aligned}
$$

- Same diagonal structure as homogeneous heat equation?

Explicit Formulation

$$\begin{aligned}\frac{I_{x,y}^{t+1} - I_{x,y}^{t}}{\lambda} &= c_{x,y}(I_{x-1,y}^{t} + I_{x+1,y}^{t} + I_{x,y-1}^{t} + I_{x,y+1}^{t} - 4I_{x,y}^{t}) \\ &+ (\frac{c_{x+1,y} - c_{x-1,y}}{2})(\frac{I_{x+1,y} - I_{x-1,y}}{2}) \\ &+ (\frac{c_{x,y+1} - c_{x,y-1}}{2})(\frac{I_{x,y+1} - I_{x,y-1}}{2})\end{aligned}$$

- Same diagonal structure as homogeneous heat equation? Yes.
- Symmetric?

Explicit Formulation

$$
\begin{aligned}
\frac{I^{t+1}_{x,y} - I^t_{x,y}}{\lambda} &= c_{x,y}(I^t_{x-1,y} + I^t_{x+1,y} + I^t_{x,y-1} + I^t_{x,y+1} - 4I^t_{x,y}) \\
&+ (\frac{c_{x+1,y} - c_{x-1,y}}{2})(\frac{I_{x+1,y} - I_{x-1,y}}{2}) \\
&+ (\frac{c_{x,y+1} - c_{x,y-1}}{2})(\frac{I_{x,y+1} - I_{x,y-1}}{2})
\end{aligned}
$$

- Same diagonal structure as homogeneous heat equation? Yes.
- Symmetric? No.
- Diagonal dominance?

Explicit Formulation

$$
\begin{aligned}
\frac{I^{t+1}_{x,y} - I^t_{x,y}}{\lambda} &= c_{x,y}(I^t_{x-1,y} + I^t_{x+1,y} + I^t_{x,y-1} + I^t_{x,y+1} - 4I^t_{x,y}) \\
&+ (\frac{c_{x+1,y} - c_{x-1,y}}{2})(\frac{I_{x+1,y} - I_{x-1,y}}{2}) \\
&+ (\frac{c_{x,y+1} - c_{x,y-1}}{2})(\frac{I_{x,y+1} - I_{x,y-1}}{2})
\end{aligned}
$$

- Same diagonal structure as homogeneous heat equation? Yes.
- Symmetric? No.
- Diagonal dominance? Data dependent.

Outline

Explicit Formulation

How do we get from this:

$$\frac{\partial I}{\partial t} = c(x, y, t)\nabla^2 I + \nabla c \cdot \nabla I$$

to Equation 7?

By splitting the Laplacian and averaging the forward and backward differences in the gradient:

$$\begin{aligned}
\frac{I^{t+1}_{x,y} - I^{t}_{x,y}}{\lambda} &= c_{x,y}[(I^t_{x-1,y} - I^t_{x,y}) + (I^t_{x+1,y} - I^t_{x,y}) \\
&+ (I^t_{x,y-1} - I^t_{x,y}) + (I^t_{x,y+1} - I^t_{x,y})] \\
&+ \frac{\partial c}{\partial x}[\frac{I_{x+1,y} - I_{x,y}}{2} + \frac{I_{x,y} - I_{x-1,y}}{2}] \\
&+ \frac{\partial c}{\partial y}[\frac{I_{x,y+1} - I_{x,y}}{2} + \frac{I_{x,y} - I_{x,y-1}}{2}]
\end{aligned}$$

Explicit Formulation

$$\frac{\partial I}{\partial t} = c(x,y,t)\nabla^2 I + \nabla c \cdot \nabla I$$

$$\begin{aligned}
\frac{I_{x,y}^{t+1} - I_{x,y}^{t}}{\lambda} &= (c_{x,y} - \frac{1}{2}\frac{\partial c}{\partial x})(I_{x-1,y}^{t} - I_{x,y}^{t}) \\
&+ (c_{x,y} + \frac{1}{2}\frac{\partial c}{\partial x})(I_{x+1,y}^{t} - I_{x,y}^{t}) \\
&+ (c_{x,y} - \frac{1}{2}\frac{\partial c}{\partial y})(I_{x,y-1}^{t} - I_{x,y}^{t}) \\
&+ (c_{x,y} + \frac{1}{2}\frac{\partial c}{\partial y})(I_{x,y+1}^{t} - I_{x,y}^{t})
\end{aligned}$$

Explicit Formulation

These are first order Taylor series approximations

$$c_{x,y} + \frac{1}{2}\frac{\partial c}{\partial x} \approx c_{x+\frac{1}{2},y}$$

$$c_{x,y} - \frac{1}{2}\frac{\partial c}{\partial x} \approx c_{x-\frac{1}{2},y}$$

$$c_{x+\frac{1}{2},y} \approx g\left(\frac{s_{x,y} + s_{x+1,y}}{2}\right)$$

$$c_{x-\frac{1}{2},y} \approx g\left(\frac{s_{x,y} + s_{x-1,y}}{2}\right)$$

Where $s_{x,y} = ||\nabla I(x,y)||$.

Explicit Formulation

$$
\begin{aligned}
\frac{I_{x,y}^{t+1} - I_{x,y}^{t}}{\lambda} &= g\left(\frac{s_{x,y} + s_{x-1,y}}{2}\right)\left(I_{x-1,y}^{t} - I_{x,y}^{t}\right) \\
&+ g\left(\frac{s_{x,y} + s_{x+1,y}}{2}\right)\left(I_{x+1,y}^{t} - I_{x,y}^{t}\right) \\
&+ g\left(\frac{s_{x,y} + s_{x,y-1}}{2}\right)\left(I_{x,y-1}^{t} - I_{x,y}^{t}\right) \\
&+ g\left(\frac{s_{x,y} + s_{x,y+1}}{2}\right)\left(I_{x,y+1}^{t} - I_{x,y}^{t}\right)
\end{aligned}
$$

Anisotropic Implementation

Compute $g()$ using the projection of the gradient along one direction. For example, in $g(\frac{s_{x,y}+s_{x+1,y}}{2})$, let

$$
\begin{aligned}
s_{x,y} &= ||\frac{\partial I}{\partial x}(x,y)|| \\
s_{x+1,y} &= ||\frac{\partial I}{\partial x}(x+1,y)||
\end{aligned}
$$

Computing $s_{x,y}$ using forward differences, and $s_{x+1,y}$ using backward differences

$$
\begin{aligned}
s_{x,y} &= ||I_{x+1,y} - I_{x,y}|| \\
s_{x+1,y} &= ||I_{x+1,y} - I_{x,y}||,
\end{aligned}
$$

so $g(\frac{s_{x,y}+s_{x+1,y}}{2}) = g(||I(x+1,y) - I(x,y)||)$.

Explicit Formulation

$$
\begin{aligned}
\frac{I_{x,y}^{t+1} - I_{x,y}^{t}}{\lambda} &= g(|I_{x-1,y} - I_{x,y}|)(I_{x-1,y}^{t} - I_{x,y}^{t}) \\
&+ g(|I_{x+1,y} - I_{x,y}|)(I_{x+1,y}^{t} - I_{x,y}^{t}) \\
&+ g(|I_{x,y-1} - I_{x,y}|)(I_{x,y-1}^{t} - I_{x,y}^{t}) \\
&+ g(|I_{x,y+1} - I_{x,y}|)(I_{x,y+1}^{t} - I_{x,y}^{t})
\end{aligned}
$$

Notation:

The authors use $\triangledown$ to denote finite differences. This is not the gradient operator (∇).

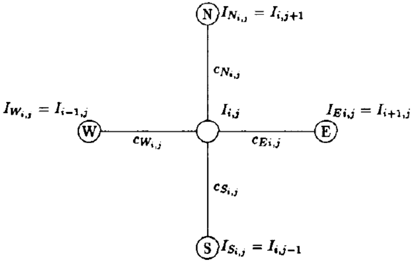


Image neighborhood system

$$
\begin{aligned}
\nabla_N I_{i,j} &\equiv I_{i-1,j} - I_{i,j} \\
\nabla_S I_{i,j} &\equiv I_{i+1,j} - I_{i,j} \\
\nabla_E I_{i,j} &\equiv I_{i,j+1} - I_{i,j} \\
\nabla_W I_{i,j} &\equiv I_{i,j-1} - I_{i,j}
\end{aligned}
$$

$$
\begin{aligned}
c_{N_{i,j}} &= g(|\nabla_N I_{i,j}|) \\
c_{S_{i,j}} &= g(|\nabla_S I_{i,j}|) \\
c_{E_{i,j}} &= g(|\nabla_E I_{i,j}|) \\
c_{W_{i,j}} &= g(|\nabla_W I_{i,j}|)
\end{aligned}
$$

Explicit Formulation

The previous explicit formulation

$$\begin{aligned}\frac{I_{x,y}^{t+1} - I_{x,y}^{t}}{\lambda} &= g(|I_{x-1,y} - I_{x,y}|)(I_{x-1,y}^{t} - I_{x,y}^{t}) \\ &+ g(|I_{x+1,y} - I_{x,y}|)(I_{x+1,y}^{t} - I_{x,y}^{t}) \\ &+ g(|I_{x,y-1} - I_{x,y}|)(I_{x,y-1}^{t} - I_{x,y}^{t}) \\ &+ g(|I_{x,y+1} - I_{x,y}|)(I_{x,y+1}^{t} - I_{x,y}^{t})\end{aligned}$$

can be rewritten as

$$\begin{aligned}I_{x,y}^{t+1} &= I_{x,y}^{t} + \lambda(c_{N_{i,j}} \bigtriangledown_N I_{i,j} + c_{S_{i,j}} \bigtriangledown_S I_{i,j} \\ &+ c_{E_{i,j}} \bigtriangledown_E I_{i,j} + c_{W_{i,j}} \bigtriangledown_W I_{i,j})^{t}\end{aligned}$$

Perona-Malik Implementation

The implementation looks like a discretization of anisotropic diffusion with a diagonal diffusion tensor.

$$
\begin{aligned}
\partial_t u &= \operatorname{div}\left(\begin{bmatrix} c_E u_x \\ c_N u_y \end{bmatrix} + \begin{bmatrix} c_W u_x \\ c_S u_y \end{bmatrix} \right) \\
&= \operatorname{div}\left(\begin{bmatrix} c_E & 0 \\ 0 & c_N \end{bmatrix} \nabla u + \begin{bmatrix} c_W & 0 \\ 0 & c_S \end{bmatrix} \nabla u \right)
\end{aligned}
$$

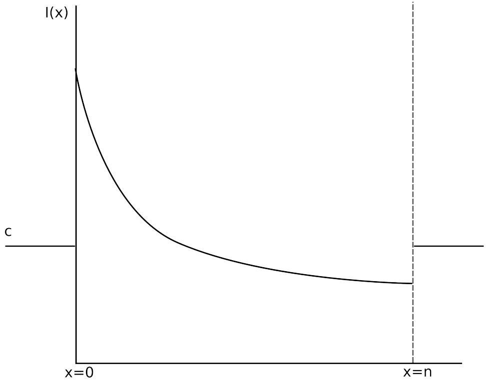
Discrete Maximum Principle

It can be shown that

- The algorithm will not lead to the production of new local maxima.
- Similarly, no new local minima will be created.
- Therefore, the Perona-Malik algorithm can be used to create scale-space image representations.

Outline

Constant Boundary Value

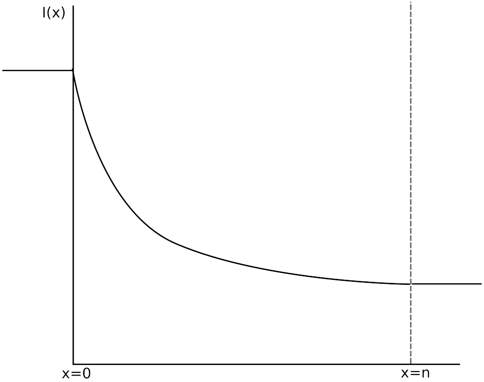


$(x < 0)$ or $(x > n) \rightarrow I(x) = c$

For $c = 0$:

$$I_{xx}(0) \approx -2I(0) + I(1)$$

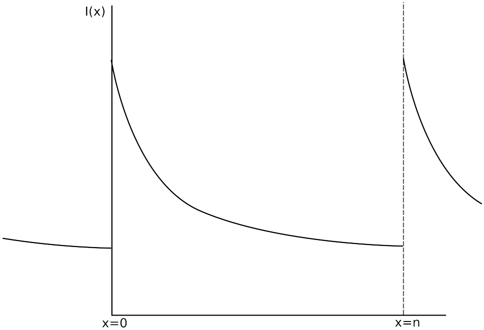
Constant Boundary Slope



Fixing the slope at zero (adiabatic) gives
$(x < 0) \rightarrow I(x) = I(0)$
$(x > n) \rightarrow I(x) = I(n)$

$$I_{xx}(0) \approx -I(0) + I(1)$$

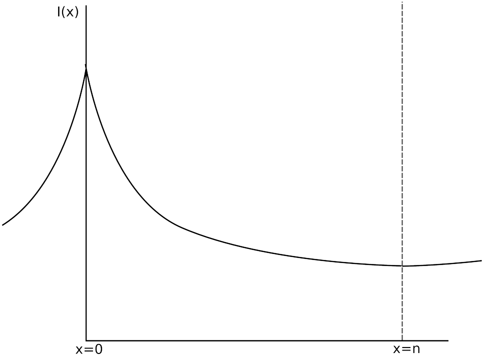
Periodic Boundary Conditions



$$(x < 0) \rightarrow I(x) = I(x+n)$$
$$(x > n) \rightarrow I(x) = I(x-n)$$

$$I_{xx}(0) \approx I(n-1) - 2I(0) + I(1)$$

Reflective Boundary Conditions



$$(x < 0) \rightarrow I(x) = I(-x)$$
$$(x > n) \rightarrow I(x) = I(2n - x)$$

$$I_{xx}(0) \approx -2I(0) + 2I(1)$$

Edge Enhancement

Inhomogeneous diffusion may actually enhance edges, for a certain choice of $c(x, y, t)$.

1D example:

Let $s(x) = \frac{\partial I}{\partial x}$, and $\phi(s) = g(I_x)I_x = g(s)s$.

The 1D inhomogeneous heat equation becomes

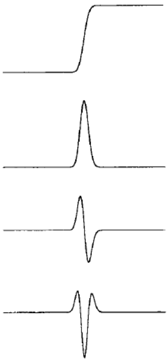
$$
\begin{aligned}
I_t = \frac{\partial}{\partial x}(g(I_x)I_x) &= \frac{\partial}{\partial x}\phi(s(x)) \\
\text{by chain rule} &= \frac{\partial \phi}{\partial s}\frac{\partial s}{\partial x} \\
I_t &= \phi'(s(x))I_{xx}
\end{aligned}
$$

Edge Enhancement

We are interested in the rate of change of edge slope with respect to time.

$$\begin{aligned}
\frac{\partial}{\partial t}(I_x) &= \frac{\partial}{\partial x}(I_t) \quad \text{if I is smooth} \\
&= \frac{\partial}{\partial x}(\phi'(s) I_{xx}) \\
&= \phi''(s)\frac{\partial s}{\partial x} I_{xx} + \phi'(s) I_{xxx} \\
&= \phi''(s) I_{xx}^2 + \phi'(s) I_{xxx}
\end{aligned}$$

Edge Enhancement



I, I_x, I_{xx}, I_{xxx}

$$\frac{\partial}{\partial t}(I_x) = \phi''(s(x))I_{xx}^2 + \phi'(s(x))I_{xxx}$$

For a step edge with $I_x > 0$ look at the inflection point, p, where the slope is maximum.

Observe that $I_{xx}(p) = 0$, and $I_{xxx}(p) < 0$.

$$\frac{\partial}{\partial t}(I_x)(p) = \phi'(s(p))I_{xxx}(p)$$

The sign of this quantity depends only on $\phi'(s(p))$.

Edge Enhancement

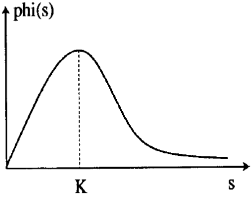
At the inflection point:

$$\frac{\partial}{\partial t}(I_x)(p) = \phi'(s)I_{xxx}(p)$$

- If $\phi'(s) > 0$, then $\frac{\partial}{\partial t}(I_x)(p) < 0$ (slope is decreasing).
- If $\phi'(s) < 0$, then $\frac{\partial}{\partial t}(I_x)(p) > 0$ (slope is increasing).

Since $\phi(s) = g(s)s$, selecting the function $g(s)$ determines which edges are smoothed and which are sharpened.

The function $\phi(s) = g(s)s$



- $\phi(0) = 0$
- $\phi'(s) > 0$ for $s < K$
- $\phi'(s) < 0$ for $s > K$
- $\lim_{s\to\infty} \phi(s) \to 0$