# Medical Image Analysis

CS 778 / 578

Computer Science and Electrical Engineering Dept.
West Virginia University

February 4, 2011

# Outline


1. Norms
2. TV-norm
3. TV-norm minimization
4. Implementation
5. Fixed point methods
6. Results

# Outline


1. Norms
2. TV-norm
3. TV-norm minimization
4. Implementation
5. Fixed point methods
6. Results

# Vector and function norms

- $||v||_1 = |x| + |y| + |z|$
- $||v||_2 = \sqrt{x^2 + y^2 + z^2}$
- $||v||_\infty = \max(|x|, |y|, |z|)$

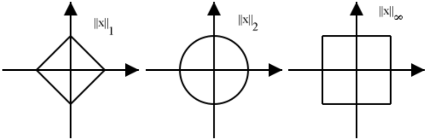



Figure: Points equidistant from the origin under various norms.

The generalized vector p-norm is $(\sum_i |x_i|^p)^{1/p}$

**The p-norm of the function, $f(x)$ is given by**

$$L_p(f) = \left[\int_\Omega |f(x)|^p dx\right]^{1/p} \quad (1)$$

# Outline


1. Norms
2. TV-norm
   - Properties
3. TV-norm minimization
4. Implementation
5. Fixed point methods
6. Results

# Definition

The TV-norm is the L1 functional norm of the gradient magnitude.

$$TV(u) = \int_{\Omega} ||\nabla u|| \, dx \, dy$$

**Recall**

Minimizing the L2 functional norm of the gradient magnitude led to the isotropic heat equation.

# Membrane Spline vs Total Variation

The **membrane spline** energy functional

$$\int_{\Omega} ||\nabla u||^2 \, dx \, dy$$

represents the elastic potential energy of a thin sheet of material.

The **total variation** energy functional

$$\int_{\Omega} ||\nabla u|| \, dx \, dy$$

represents the oscillation of the function $u$.

# Membrane Spline vs Total Variation

Note that

$$\left[\int_\Omega |f(x)|^p dx\right]^{1/p}$$

has the same minimizer as

$$\int_\Omega |f(x)|^p dx.$$

(Since $f(x)$ is always positive and $g(x) = x^{1/p}$ is monotonic.)

# Membrane Spline vs Total Variation

The **membrane spline** energy functional

$$\int_{\Omega} ||\nabla u||^2 \, dx \, dy$$

has the same minimizer as $L_2(||\nabla u||)$.

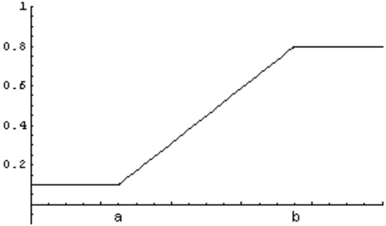
The **total variation** energy functional

$$\int_{\Omega} ||\nabla u|| \, dx \, dy$$

is equivalent to $L_1(||\nabla u||)$.

# Total Variation Does Not Penalize Discontinuities

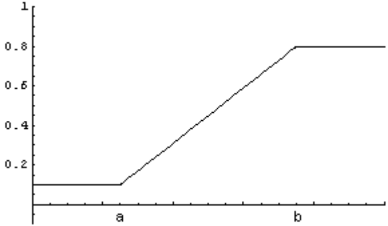
Consider a function, $u$, with a step of height $h$:




$$
\begin{aligned}
MEM(f) &= \int_{\Omega} u_x^2\,dx = \int_a^b \left(\frac{h}{b-a}\right)^2 dx \\
&= \left(\frac{h}{b-a}\right)^2 \int_a^b dx = \left(\frac{h}{b-a}\right)^2 (b-a) \\
&= \frac{h^2}{b-a}
\end{aligned}
$$

# Total Variation Does Not Penalize Discontinuities

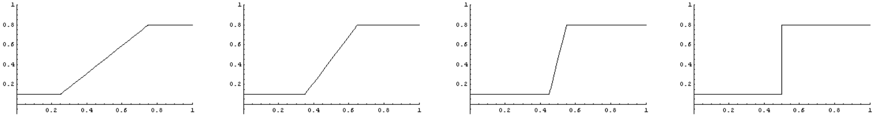
Consider a function, $u$, with a step of height $h$:




$$
\begin{aligned}
TV(f) &= \int_\Omega \sqrt{u_x^2}\,dx = \int_a^b |\frac{h}{b-a}|\,dx \\
&= |\frac{h}{b-a}| \int_a^b dx = |\frac{h}{b-a}|(b-a) \\
&= |h|
\end{aligned}
$$

# Total Variation Does Not Penalize Discontinuities

$$MEM(f) = \frac{h^2}{b-a}, TV(f) = |h|$$



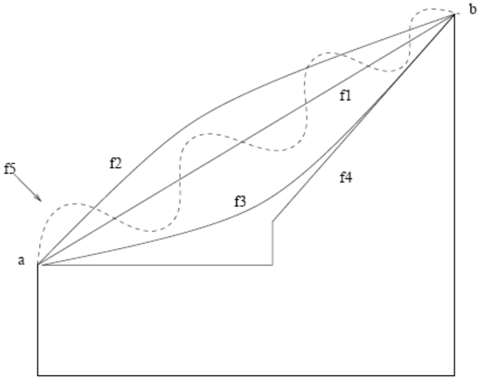

MEM(f1) = 0.98
TV(f1) = 0.7

MEM(f2) = 1.63
TV(f2) = 0.7

MEM(f3) = 4.9
TV(f3) = 0.7

MEM(f4) = $\infty$
TV(f4) = 0.7

# Total Variation Does Penalize Oscillation




$$TV(f_5) > TV(f_1) = TV(f_2) = TV(f_3) = TV(f_4)$$

# Outline

1. Norms
2. TV-norm
3. TV-norm minimization
4. Implementation
5. Fixed point methods
6. Results

# Variational Calculus

Minimizing the TV-norm functional

$$E(u) = \int_\Omega ||\nabla u||\, dx = \int_\Omega \sqrt{u_x^2 + u_y^2}\, dx$$

has Euler-Lagrange condition for minimization:

$$\begin{aligned} \nabla E &= f_u - \frac{\partial}{\partial x} f_{u_x} - \frac{\partial}{\partial y} f_{u_y} = 0 \\ &= 0 - \frac{\partial}{\partial x}(u_x(u_x^2 + u_y^2)^{-\frac{1}{2}}) - \frac{\partial}{\partial y}(u_y(u_x^2 + u_y^2)^{-\frac{1}{2}}) = 0 \end{aligned}$$

Leading to the descent equation

$$\partial_t u = \mathrm{div}(\frac{\nabla u}{||\nabla u||})$$

# Energy Minimization

Membrane spline energy

$$\min_u \int_\Omega u_x^2 + u_y^2 \, dx \, dy$$

Has Euler-Lagrange condition for minimization given by

$$\text{div}(\nabla u) = 0$$

and descent equation

$$\partial_t u = \text{div}(\nabla u)$$

Total Variation

$$\min_u \int_\Omega \sqrt{u_x^2 + u_y^2} \, dx \, dy$$

Has Euler-Lagrange condition for minimization given by

$$\text{div}\left(\frac{\nabla u}{||\nabla u||}\right) = 0$$

and descent equation

$$\partial_t u = \text{div}\left(\frac{\nabla u}{||\nabla u||}\right)$$

# Descent Equation

$$\partial_t u = \text{div}(\frac{\nabla u}{||\nabla u||})$$

This is an inhomogeneous diffusion equation with diffusivity

$$g(||\nabla u||) = \frac{1}{||\nabla u||}$$

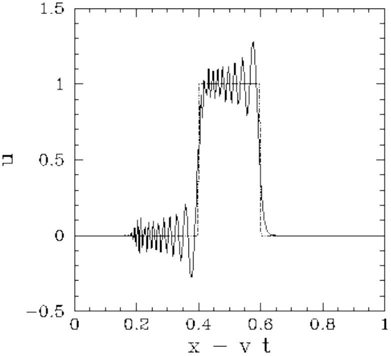
- Diffusion is slowed for large image gradients.
- Diffusion is speeded up for small image gradients.
- Divergence is due only to changing gradient directions.

# Outline


1. Norms
2. TV-norm
3. TV-norm minimization
4. Implementation
   - Upwind finite differences
   - $\epsilon$ regularization
5. Fixed point methods
6. Results

# Upwind finite differences

Since we are allowing steep discontinuities, numerical differentiation can be a source of instability.




Solution : a slope limiting derivative operator.

Denote the forward finite difference of $u$ at $(i,j)$ in the x direction by:

$$\Delta_+^x u_{ij} = u_{i+1,j} - u_{i,j}$$

and the backward difference by:

$$\Delta_-^x u_{ij} = u_{i,j} - u_{i-1,j}$$

We will denote the upwind finite difference in the x direction as:

$$\text{minmod}(\Delta_+^x u_{ij}, \Delta_-^x u_{ij})$$

# Minmod operation

$$\text{minmod}(a, b) = \left(\frac{\text{sgn}\, a + \text{sgn}\, b}{2}\right) \min(|a|, |b|)$$

where

$$\text{sgn}\, x = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$$

# Minmod operation

$$\text{minmod}(a,b) = \left(\frac{\operatorname{sgn} a + \operatorname{sgn} b}{2}\right) \min(|a|,|b|)$$

- if a or b are 0, $\text{minmod}(a,b) = 0$
- if a and b have opposite sign, $\text{minmod}(a,b) = 0$
- otherwise, pick the one with the smallest magnitude.

# What happens when $||\nabla u|| \to 0$

Singularities may develop in very smooth regions.

$$\partial_t u = \text{div}\left(\frac{\nabla u}{||\nabla u|| + \epsilon}\right)$$

- One strategy: use fixed constant $\epsilon \ll 1$.
- Another strategy: relaxation. Start with a moderate $\epsilon$ and decrease over time.

See "Nonlinear Total Variation Based Noise Removal Algorithms.", L. Rudin, S. Osher, E. Fatemi, Physica D, 1992.

# Outline

1. Norms
2. TV-norm
3. TV-norm minimization
4. Implementation
5. Fixed point methods
6. Results

# Another approach to minimization

Rather than solving the descent equation, we may try to directly solve the minimization condition

$$0 = \mathrm{div}\left(\frac{\nabla u}{||\nabla u||}\right) - \beta(u^0 - u)$$

A fixed-point iteration scheme is obtained by assigning iteration numbers to $u$, such as $u^k$ and $u^{k+1}$. Note that there is no longer any time step parameter. The system is iterated until convergence is achieved.

# Lagged-diffusivity fixed-point

One possible method for assigning iteration numbers to the equation is

$$0 = \mathrm{div}\left(\frac{\nabla u^{k+1}}{||\nabla u^k||}\right) - \beta(u^0 - u^{k+1})$$

This assignment results in a linear system. Since the diffusivity is computed based on the previous iteration, this is known as the **lagged-diffusivity** fixed-point iteration.

# Lagged-diffusivity fixed-point

The equation can be discretized and a linear system in the following form can be obtained

$$Bu^{k+1} = c(u^0, u^k)$$

Solving for $u^{k+1}$ represents a single fixed-point iteration.

See also:

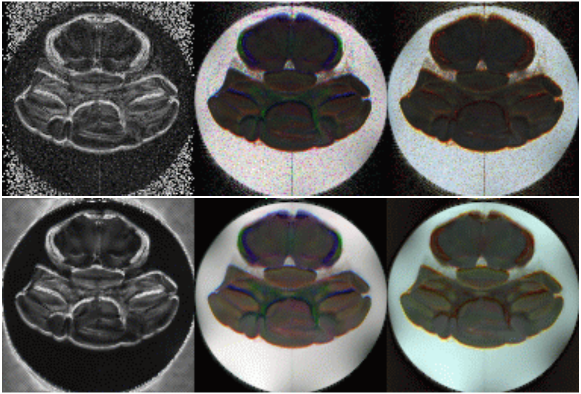
"Iterative Methods for Total Variation Denoising", Vogel, C.R. and Oman, M.E., SIAM JOURNAL ON SCIENTIFIC COMPUTING, vol. 17, pp. 227–238,1996.

"On the Convergence of the Lagged Diffusivity Fixed Point Method in Total Variation Image Restoration" Chan, T.F. and Mulet, P., SIAM Journal on Numerical Analysis, vol. 36, no. 2, pp. 354–367, 1999.

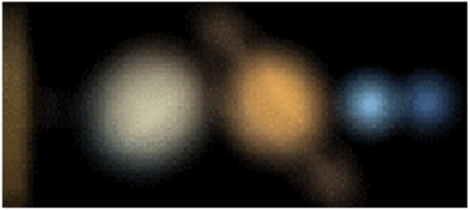
# Outline


1. Norms
2. TV-norm
3. TV-norm minimization
4. Implementation
5. Fixed point methods
6. Results
   - Drawbacks

# Denoising Results



# Deconvolution Results



Since TV-norm allows sharp edges to form, it is a good side constraint for deconvolution.

# Deconvolution Formulation

The convolution $k * I$ can be written as the matrix-vector multiplication $Ku$ where u is the vectorized image, and K is a matrix whose diagonals contain the convolution kernel values.

The data constraint then becomes

$$min_u \int_\Omega \frac{1}{2}(Ku - u^0)^2 \, dx$$

The corresponding condition for minimization is

$$K(Ku - u^0)^2 = 0$$

See "Total variation blind deconvolution", Chan, TF and Wong, C.K., IEEE Transactions on Image Processing, vol. 7, no. 3, pp. 370–375, 1998.

# A Drawback

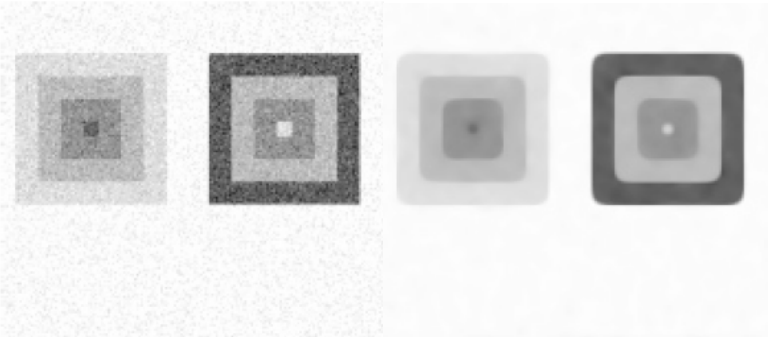
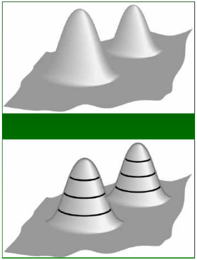


Image contours may be rounded.

# Image Contours

Another geometric interpretation of TV norm minimization: Consider isocontours of the image (the curves of constant image intensity)



- Evolution of the image $u$ is also evolution of the isocontours, $c$.
- TV-norm minimization is smoothing of the isocontours of $u$.