Medical Image Analysis

CS 778 / 578

Computer Science and Electrical Engineering Dept.
West Virginia University

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Outline

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Our assumption has been that images are characterized by

- Piecewise smooth regions
- separated by edges

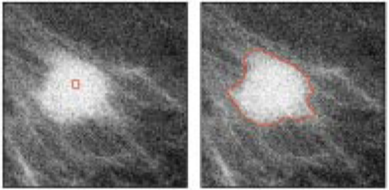
So far, we have concentrated on finding the edges and designing a speed function which emphasizes the edges we are interested in.

Now we will concentrate, instead, on identifying the homogeneous regions.

- It may be easy to identify these regions when noise is present.
- Our image model may not hold : there may be no discernable edges between regions.

Active Contours

Without Edges : Objects may not have edges defined by the image gradient.



This is a **region-based** segmentation technique. Evolution of the curve is governed by properties of the region of $I(x, y)$ enclosed by the curve.

Chan and Vese present a level set implementation for minimizing a special case of the Mumford-Shah functional.

Mumford-Shah Functional

Let $u_0(x, y)$ be the original image, and $u(x, y)$ be some model for the image.

$$
\begin{aligned}
E_{MS}(u, c) &= \mu \int_0^1 ||c'(s)||ds \\
&+ \lambda \int_\Omega |u_0(x, y) - u(x, y)|^2 dx\, dy \\
&+ \int_{\Omega/c} ||\nabla u(x, y)||^2 dx\, dy
\end{aligned}
$$

- Term 1: Smooth boundary curve
- Term 2: Model fit error
- Term 3: Smooth u(x,y) - except on boundary

Mumford-Shah Functional

Piecewise constant model fitting term

$$
\begin{aligned}
u(x,y) &= c_1 \text{ inside c(s)} \\
&= c_2 \text{ outside c(s)}
\end{aligned}
$$

so, $|\nabla u(x,y)| = 0$.

$$
\begin{aligned}
E_{MS}(c_1, c_2, c) &= \mu \int_0^1 ||c'(s)||ds \\
&+ \lambda_1 \int_{inside(c)} |u_0(x,y) - c_1|^2 dx\, dy \\
&+ \lambda_2 \int_{outside(c)} |u_0(x,y) - c_2|^2 dx\, dy
\end{aligned}
$$

Mumford-Shah Functional

Piecewise constant model

$$\min_c \sum_{i=1}^{n} (c - x_i)^2$$

$$\begin{aligned} \frac{d}{dc} \sum_{i=1}^{n} (c - x_i)^2 &= \sum_{i=1}^{n} 2(c - x_i) = 0 \\ &= 2(nc - \sum_{i=1}^{n} x_i) = 0 \end{aligned}$$

So,

$$c = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$c =$ arithmetic mean of x_i

Mumford-Shah Functional

$$
\begin{aligned}
E_{MS}(c_1, c_2, c) &= \mu \int_0^1 ||c'(s)||ds \\
&+ \lambda_1 \int_{inside(c)} |u_0(x,y) - c_1|^2 dx\, dy \\
&+ \lambda_2 \int_{outside(c)} |u_0(x,y) - c_2|^2 dx\, dy
\end{aligned}
$$

where c_1 is the mean of u_0 inside the curve c, and
c_2 is the mean of u_0 outside the curve c.

With a level set approach, it is simple to determine which voxels are inside/outside by checking the sign of ψ.

Evolution Equation

In the Chan-Vese paper, the energy functional is written in level-set form in terms of heaviside functions, $H(z)$ and delta functions, $\delta(z)$.
Let $\phi(x, y)$ be the embedding function for the curve defined so that $\phi > 0$ inside of the curve.

$$u(x, y) = c_1 H(\phi(x, y)) + c_2(1 - H(\phi(x, y)))$$

and $c_1 = \text{mean}(u_0)$ in $\{\phi \geq 0\}$

$$c_1 = \frac{\int_\Omega u_0(x, y) H(\phi(x, y)) dx\, dy}{\int_\Omega H(\phi(x, y)) dx\, dy}$$

and $c_2 = \text{mean}(u_0)$ in $\{\phi < 0\}$

$$c_2 = \frac{\int_\Omega u_0(x, y)(1 - H(\phi(x, y))) dx\, dy}{\int_\Omega (1 - H(\phi(x, y))) dx\, dy}$$

Chan-Vese Functional

Length of boundary $\{\phi = 0\}$

$$\begin{aligned} L &= \int_{\Omega} ||\nabla H(\phi(x,y))|| dx\, dy \\ &= \int_{\Omega} \delta(\phi(x,y)) ||\nabla \phi(x,y)|| dx\, dy \end{aligned}$$

Chan-Vese Functional

Model Fitting Terms

$$\begin{aligned}&\int_{\phi>0} |u_0(x,y) - c_1|^2 dx\,dy\\ = &\int_{\Omega} |u_0(x,y) - c_1|^2 H(\phi(x,y)) dx\,dy\end{aligned}$$

$$\begin{aligned}&\int_{\phi<0} |u_0(x,y) - c_2|^2 dx\,dy\\ = &\int_{\Omega} |u_0(x,y) - c_2|^2 (1 - H(\phi(x,y))) dx\,dy\end{aligned}$$

Chan-Vese Functional

$$
\begin{aligned}
E(c_1, c_2, \phi) &= \mu \int_\Omega \delta(\phi(x,y)) |\nabla \phi(x,y)| \, dx\, dy \\
&+ \lambda_1 \int_\Omega |u_0 - c_1|^2 H(\phi(x,y)) \, dx\, dy \\
&+ \lambda_2 \int_\Omega |u_0 - c_2|^2 (1 - H(\phi(x,y))) \, dx\, dy
\end{aligned}
$$

This leads to the level set evolution equation

$$
\frac{d\phi}{dt} = \delta(\phi) \left[\mu \, \mathrm{div}\left(\frac{\nabla \phi}{||\nabla \phi||}\right) - \lambda_1 (u_0 - c_1)^2 + \lambda_2 (u_0 - c_2)^2 \right]
$$

Chan-Vese Evolution Equation

Use regularized $H(z)$ and $\delta(z)$ functions.
Use same discretized divergence operator as in Rudin-Osher-Fatemi TV norm paper.

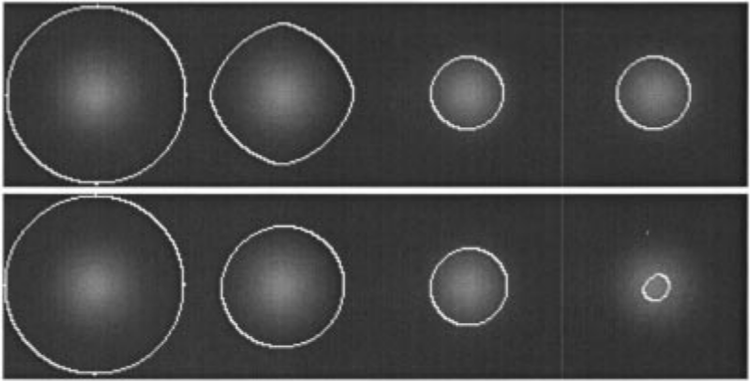
1. Initialize ϕ
2. Compute $c_1(\phi^n)$ and $c_2(\phi^n)$.
3. Compute ϕ^{n+1}
4. If solution is not stationary, go to 1.

Chan-Vese Implementation

$$
\begin{aligned}
\frac{\phi^{n+1} - \phi^n}{\Delta t} &= \delta_h(\phi^n)[\mu \Delta_-^x (\frac{\Delta_+^x \phi^{n+1}}{\sqrt{(\Delta_+^x \phi^n)^2 + (\Delta_c^y \phi^n)^2}}) \\
&+ \mu \Delta_-^y (\frac{\Delta_+^y \phi^{n+1}}{\sqrt{(\Delta_c^x \phi^n)^2 + (\Delta_+^y \phi^n)^2}})) \\
&- \lambda_1 (u_0 - c_1)^2 + \lambda_2 (u_0 - c_2)^2]
\end{aligned}
$$

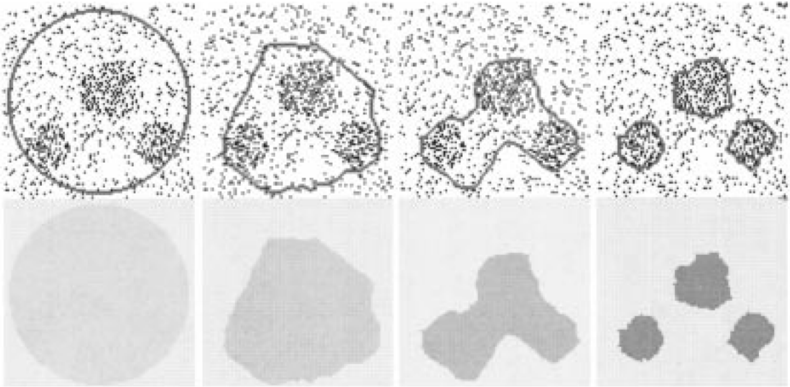
Outline

Chan-Vese Results



Region-based (top) and edge-based (bottom) segmentation results.

Chan-Vese Results



Piecewise smooth

Let $u_0(x, y)$ be the original image, and $u(x, y)$ be some model for the image.

$$
\begin{aligned}
E_{MS}(u, c) &= \mu \int_0^1 ||c'(s)||ds \\
&+ \lambda \int_\Omega |u_0(x, y) - u(x, y)|^2 dx\, dy \\
&+ \int_{\Omega/c} |\nabla u(x, y)|^2 dx\, dy
\end{aligned}
$$

- Term 1: Smooth boundary curve, c
- Term 2: Model fit error
- Term 3: Smooth u(x,y), except possibly at c

Tsai-Yezzi Piecewise Smooth Results

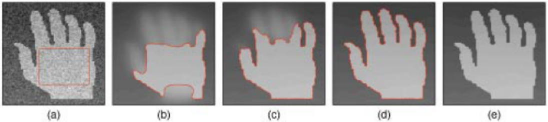


Fig. 2. Outward flow from inside.

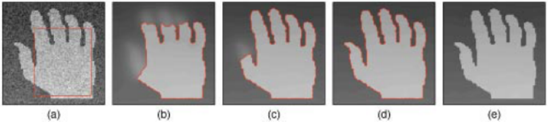


Fig. 3. Bidirectional flow.

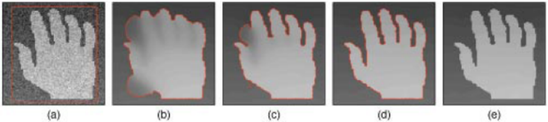
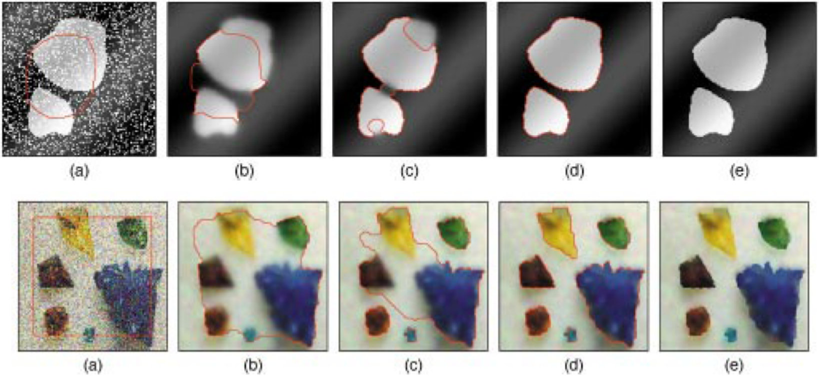


Fig. 4. Inward flow from outside.

Simultaneous Restoration and Segmentation

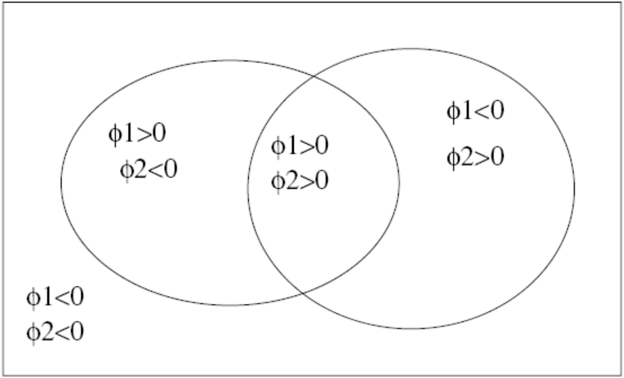


Multiphase Chan-Vese

Supports more than 2 regions by using multiple embedding functions.

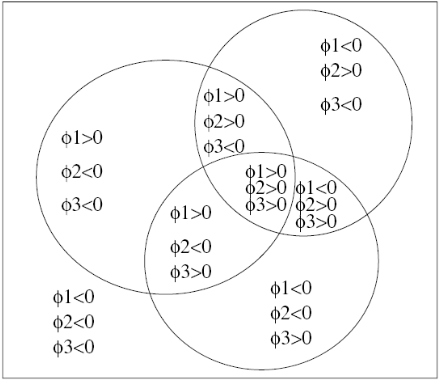
”A Multiphase Level Set Framework for Image Segmentation Using the Mumford and Shah Model”, Vese, L.A. and Chan, T.F.,International Journal of Computer Vision, vol. 50, no. 3, pp. 271-293, 2002.

Multiphase Chan-Vese



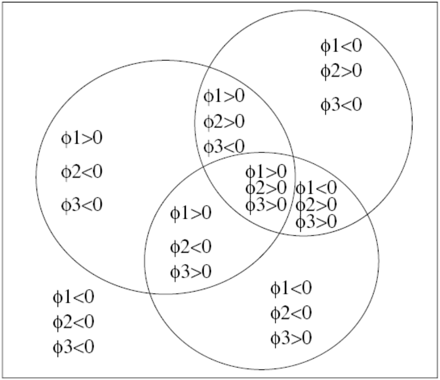
- Region 1 : $\phi_1 > 0$ and $\phi_2 > 0$
- Region 2 : $\phi_1 > 0$ and $\phi_2 < 0$
- Region 3 : $\phi_1 < 0$ and $\phi_2 > 0$
- Region 4 : $\phi_1 < 0$ and $\phi_2 < 0$

Multiphase Chan-Vese



Extends to more regions by adding new level set functions.

Multiphase Chan-Vese



Extends to more regions by adding new level set functions.

Multiphase Chan-Vese Energy Functional

$$
\begin{aligned}
E_{c,\phi_1,\phi_2} &= \mu \int_\Omega ||\nabla H(\phi_1)|| dx\, dy + \mu \int_\Omega ||\nabla H(\phi_2)|| dx\, dy \\
&+ \int_\Omega (u_0 - c_{11})^2 H(\phi_1) H(\phi_2) dx\, dy \\
&+ \int_\Omega (u_0 - c_{10})^2 H(\phi_1)(1 - H(\phi_2)) dx\, dy \\
&+ \int_\Omega (u_0 - c_{01})^2 (1 - H(\phi_1)) H(\phi_2) dx\, dy \\
&+ \int_\Omega (u_0 - c_{00})^2 (1 - H(\phi_1))(1 - H(\phi_2)) dx\, dy
\end{aligned}
$$

Multiphase Chan-Vese Evolution Equations

$$
\begin{aligned}
\frac{\partial \phi_1}{\partial t} &= \delta_\epsilon(\phi_1)\{\mu \, \mathrm{div}(\frac{\nabla \phi_1}{||\nabla \phi_1||}) \\
&- [((u_0 - c_{11})^2 - (u_0 - c_{01})^2)H(\phi_2) \\
&+ ((u_0 - c_{10})^2 - (u_0 - c_{00})^2)(1 - H(\phi_2))]\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \phi_2}{\partial t} &= \delta_\epsilon(\phi_2)\{\mu \, \mathrm{div}(\frac{\nabla \phi_2}{||\nabla \phi_2||}) \\
&- [((u_0 - c_{11})^2 - (u_0 - c_{01})^2)H(\phi_1) \\
&+ ((u_0 - c_{10})^2 - (u_0 - c_{00})^2)(1 - H(\phi_1))]\}
\end{aligned}
$$