Medical Image Analysis

CS 778 / 578

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West Virginia University

February 23, 2011

Outline

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Origins

Stanley Osher and James A. Sethian, ”Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations”, J. Comput. Phys., 1988.

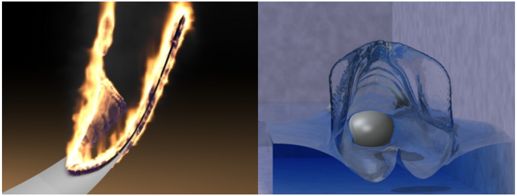
Tracking dynamic boundaries and interfaces in

- Fluid Mechanics
- Flame Propagation
- Crystallography

Especially where surfaces may split, merge, form sharp corners.

Surface Representation

- Surfaces are represented as the zero level set of an embedding function.
- This function is evolved, *implicitly* evolving the embedded curve.
- The previous (Lagrangian) approach was to track points on the interface.

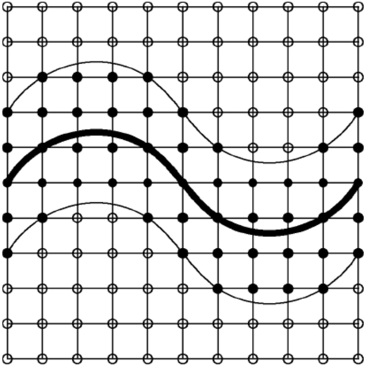


Lagrangian Approach

This was the ”snake” approach to curve evolution.

- Discretize the curve into individual particles.
- Track these particles as they move through a field.

Eulerian Approach



- Discretize the embedding function, $\psi(x, y)$ to represent and evolve the curve.
- Evolve $\psi(x, y)$ by updating at *fixed* grid locations.
- For simplicity, the function $\psi(x, y)$ can be discretized to have the same resolution as the image we are segmenting.

Problems with ”snakes”

The Lagrangian approach does not handle

- Splitting / merging boundaries (topological change)
- Self-intersection
- Sharp corners or other discontinuities

Level-set methods

The Eulerian approach can handle

- Splitting / merging boundaries (topological change)
- Self-intersection
- Sharp corners of other discontinuities

"Hamilton-Jacobi" type equations

$$\frac{\partial \psi}{\partial t} + F||\nabla \psi|| = 0$$

have been extensively studied under this framework, especially interfaces moving with curvature dependent speed ($F(\kappa)$).

Parametric vs. Implicit

Parametric Curve:

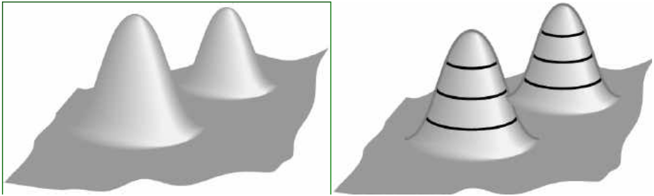
$$C(s) = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}$$

Evaluating the function gives coordinates of points on the curve.

Implicit Curve:

$$\psi(x, y) = c$$

The curve is the set of all points which satisfy the equality.



Normal and Curvature

The gradient of the embedding function is perpendicular to the level curve.

$$N(x,y) = \frac{\nabla\psi(x,y)}{||\nabla\psi(x,y)||}$$

Recall : Directional Derivative

$$D_{\mathbf{u}}f(\mathbf{p}) = \frac{d}{dh}f(\mathbf{p}+h\mathbf{u}) = \nabla f \cdot \frac{\mathbf{u}}{||\mathbf{u}||}$$

- $D_{\mathbf{u}}f(\mathbf{p})$ has the least magnitude (0) when u is parallel to the curve.
- The directions of least magnitude and greatest magnitude are perpendicular.

Normal and Curvature

The curvature of the level curve is the rate of change of the normal vector

$$\kappa = -\operatorname{div}\left(\frac{\nabla\psi}{||\nabla\psi||}\right)$$

This can be rewritten as

$$\kappa = -\frac{\psi_{xx}\psi_y^2 - 2\psi_x\psi_y\psi_{xy} + \psi_{yy}\psi_x^2}{(\psi_x^2 + \psi_y^2)^{\frac{3}{2}}}$$

Example : a circle

Consider the implicit equation for a circle

$$(x-a)^2 + (y-b)^2 = r^2$$

The gradient is

$$\nabla\psi = \begin{bmatrix} 2(x-a) \\ 2(y-b) \end{bmatrix}$$

$$||\nabla\psi|| = \sqrt{4(x-a)^2 + 4(y-b)^2} = 2\sqrt{(x-a)^2 + (y-b)^2}$$

$$\frac{\nabla\psi}{||\nabla\psi||} = \frac{1}{\sqrt{(x-a)^2 + (y-b)^2}} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

Example : a circle

Computing the curvature

$$\kappa = -\frac{\psi_{xx}\psi_y^2 - 2\psi_x\psi_y\psi_{xy} + \psi_{yy}\psi_x^2}{(\psi_x^2 + \psi_y^2)^{\frac{3}{2}}}$$

$\psi_{xx} = 2, \psi_{yy} = 2, \psi_{xy} = 0$

$$\kappa = -\frac{8(y-b)^2 + 8(x-a)^2}{(4(x-a)^2 + 4(y-b)^2)^{\frac{3}{2}}} = -\frac{1}{r}$$

Signed Distance Function

One possible embedding function for implicit curves

- $|\psi(x, y)|$ = distance from (x, y) to the curve.
- $\psi(x, y) = 0$ if (x, y) is on the curve.
- $\psi(x, y) < 0$ if (x, y) is inside the curve.
- $\psi(x, y) > 0$ if (x, y) is outside the curve.

$$||\nabla \psi|| = 1$$

Almost everywhere.

Implicit Curve Evolution

Suppose we have an evolving curve, $c(t) = [x(t), y(t)]$. Let's derive the evolution equation for for $\psi(x, y, t)$ which has $c(t)$ as a level set.

Let $c(t)$ be the zero level set of ψ so that $\psi(x(t), y(t), t) = 0$ for all t. This implies that $\frac{d\psi}{dt}(x(t), y(t), t) = 0$.
By the chain rule

$$\begin{aligned} \frac{d\psi}{dt} &= \frac{\partial\psi}{\partial x}\frac{dx}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dt} + \frac{\partial\psi}{\partial t} \\ &= (\nabla\psi \cdot c'(t)) + \frac{\partial\psi}{\partial t} \end{aligned}$$

Implicit Curve Evolution

Decompose $c'(t)$ into components tangent and normal to $c(t)$.

$$
\begin{aligned}
0 &= (\nabla\psi \cdot (v_N N(t) + v_T T(t))) + \frac{\partial\psi}{\partial t} \\
&= (\nabla\psi \cdot v_N N(t)) + \frac{\partial\psi}{\partial t}
\end{aligned}
$$

since $\nabla\psi$ is perpendicular to the tangent to $c(t)$.
Substituting the level set definition for the normal to the embedded curve

$$
0 = (\nabla\psi \cdot (v_N \frac{\nabla\psi}{||\nabla\psi||})) + \frac{\partial\psi}{\partial t}
$$

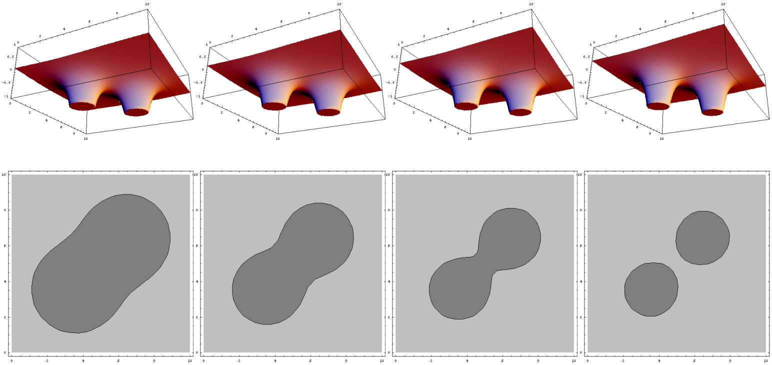
Implicit Curve Evolution

We can rewrite this result

$$
\begin{aligned}
0 &= (\nabla\psi \cdot (v_N \frac{\nabla\psi}{||\nabla\psi||})) + \frac{\partial\psi}{\partial t} \\
0 &= v_N(\nabla\psi \cdot \frac{\nabla\psi}{||\nabla\psi||}) + \frac{\partial\psi}{\partial t} \\
0 &= v_N \frac{||\nabla\psi||^2}{||\nabla\psi||} + \frac{\partial\psi}{\partial t} \\
0 &= v_N||\nabla\psi|| + \frac{\partial\psi}{\partial t}
\end{aligned}
$$

Evolving the embedding function by

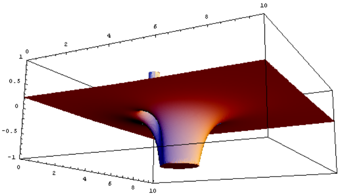
$$\frac{\partial \psi}{\partial t} = 1$$

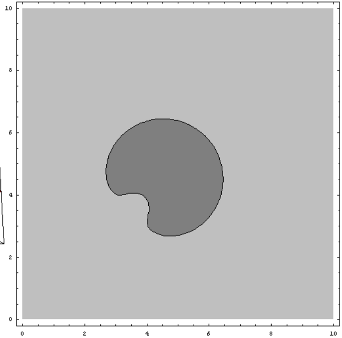


Smoothing the curve

Evolving the embedding function by

$$\frac{\partial \psi}{\partial t} = -\kappa(x, y)||\nabla \psi||$$





- $\kappa < 0$ where contour is locally convex
- $\kappa > 0$ where contour is locally concave

Curvature-Based Evolution

Curvature-Based Evolution

Letting v_N be function of curvature, we have evolution equation

$$\frac{\partial \psi}{\partial t} = -F(\kappa)||\nabla \psi||$$

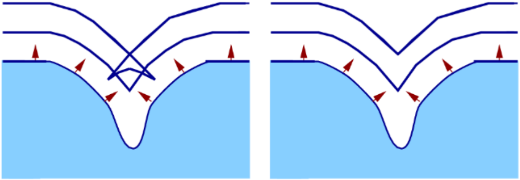
The segmentation problem is now reduced to finding an appropriate function $F(\kappa)$. We can factor $F(\kappa) = k(F_A + F_G)$ where

- F_A: the advection term is usually a constant
- F_G: depends on the geometry (curvature) of the level set
- k: is a stopping term, to slow evolution near boundaries

We can use

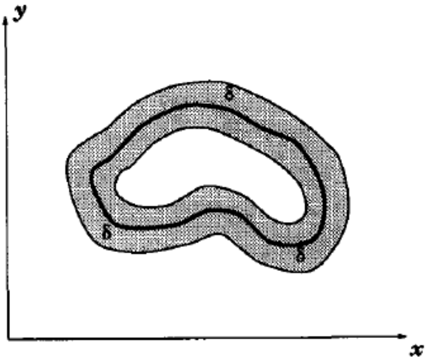
$$k(x, y) = \frac{1}{1 + ||\nabla(G_{sigma} * I(x, y))||}$$

Entropy-preserving solution



The upwind finite difference scheme we used for TV norm minimization prevents singularities, like this ”swallowtail” from developing.

Narrow-band update



For faster computation:

Since we are primarily interested in the zero level set of ψ, we may evolve only in a small region surrounding the level set.

Outline

Approach to segmentation

The curve will evolve with

- an inflation force, to reach protrusions in shape
- a curvature based speed, to keep the boundary smooth
- an image based speed, to stop the curve at image boundaries

Embed the curve, $\gamma(t)$, into ψ

Initialize ψ to be the **signed distance** to $\gamma(t=0)$.

$$\psi(x, y, t=0) = \pm d$$

where d is the distance from (x, y) to $\gamma(t=0)$.

- $d < 0$ inside γ
- $d > 0$ outside γ

Various methods

- Simple geometry, such as a circle : $\psi = \sqrt{x^2 + y^2} - r$
- Matlab `bwdist`.
- $\psi^{t+1} = \psi^t + sgn(\psi_0)(1 - ||\nabla\psi||)$

Recall

For evolving curve, $c(t)$, and embedding function $\psi(x, y, t)$:

Level set equation

$$\frac{\partial \psi}{\partial t} = -v_N ||\nabla \psi||$$

where v_N is the normal component of $c'(t)$.

How to use the level set framework to solve the segmentation problem?

<u>Balloon Inflation</u>

$$\frac{dc}{dt} = \alpha \mathbf{N} \rightarrow \frac{\partial \psi}{\partial t} = -\alpha ||\nabla \psi||$$

<u>Curvature-based motion</u>

$$\frac{dc}{dt} = \kappa \mathbf{N} \rightarrow \frac{\partial \psi}{\partial t} = -\kappa ||\nabla \psi||$$

<u>Combining both</u>

$$\frac{dc}{dt} = (\alpha + \epsilon\kappa) \mathbf{N} \rightarrow \frac{\partial \psi}{\partial t} = -(\alpha + \epsilon\kappa) ||\nabla \psi||$$

Speed Terms

Similar to the diffusivity functions used by Perona-Malik. Designed to stop the curve at object boundaries.

$$k(x, y) = \frac{1}{1 + ||\nabla(G_\sigma * I(x, y))||}$$

or

$$k(x, y) = \exp(-||\nabla(G_\sigma * I(x, y))||)$$

Use $k(x, y)$ to control the overall speed of evolution

$$\frac{dc}{dt} = k(\alpha + \epsilon\kappa)\mathbf{N}$$

Level-set Evolution Overview

The curve embedded in ψ can be evolved by

$$\frac{\partial \psi}{\partial t} = -F(\kappa)||\nabla \psi||$$

The segmentation problem is now reduced to finding an appropriate function $F(\kappa)$. We can factor $F(\kappa) = k(F_a + F_g(\kappa))$ where

- F_a: the advection term (inflation force)
- F_g: curvature based smoothing term
- k: is a stopping term, to slow evolution near boundaries

The paper uses $F(\kappa) = k(1 + \epsilon\kappa)$, but there are other possibilities.

Discretizing the evolution equation

The evolution equation

$$\frac{\partial \psi}{\partial t} = -k(1 + \epsilon\kappa)||\nabla\psi||$$

has the explicit discretization

$$\frac{\psi_{i,j}^{t+1} - \psi_{i,j}^{t}}{\Delta t} = -k_{i,j}||\nabla\psi_{i,j}^{t}|| - k_{i,j}(\epsilon\kappa_{i,j}^{t}||\nabla\psi_{i,j}^{t}||)$$

- The advection term can lead to singularities, so discretize it using upwind finite differences
- The curvature term can be discretized using central differences.

The upwind scheme

To approximate $||\nabla\psi||$

$$||\nabla\psi|| \approx \sqrt{\max(D_x^- \psi_{i,j}, 0)^2 + \min(D_x^+ \psi_{i,j}, 0)^2 + \max(D_y^- \psi_{i,j}, 0)^2 + \min(D_y^+ \psi_{i,j}, 0)^2}$$

where

- D_x^-: first order backward difference in x-direction
- D_x^+: first order forward difference in x-direction
- D_y^-: first order backward difference in x-direction
- D_y^+: first order forward difference in x-direction

Extending the Speed Function

$$\frac{dc}{dt} = k(\alpha + \epsilon\kappa)\mathbf{N}$$

We only want the values of k(x,y) on the zero level set ($\psi = 0$) to influence the evolution of ψ.

Using the evolution equation

$$\frac{\partial\psi}{\partial t} = -k(\alpha + \epsilon\kappa)||\nabla\psi||$$

allows $k(x, y)$ values from all of (x, y) to influence ψ.

The resulting embedding function can become badly conditioned: Level curves may collide, ψ may become very flat or very steep.

Suggested Schemes

The suggested schemes involve extension and reinitialization.

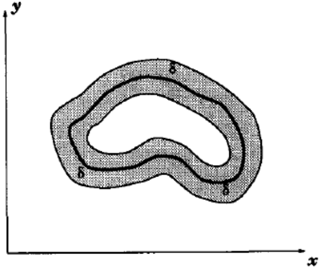
Extension of k:
Let $k(x, y) = k(x', y')$ where (x', y') is the point on the curve nearest (x, y).

Reinitialization of ψ:
Periodically recalibrate the level set by making ψ a signed distance function (SDF). The slope of the SDF is bounded almost everywhere.

Suggested Schemes

- **A**: Global extension of k over the entire domain : slow, numerical errors.
- **B**: Global extension of k with reinitialization of ψ : slow.
- **C**: Narrow-Band extension with reinitialization of ψ.

Narrow-band scheme: If ψ is a SDF, we can easily identify points near the level set. We can extend k only within this narrow band, and only evolve ψ within the narrow band.



Suggested Schemes

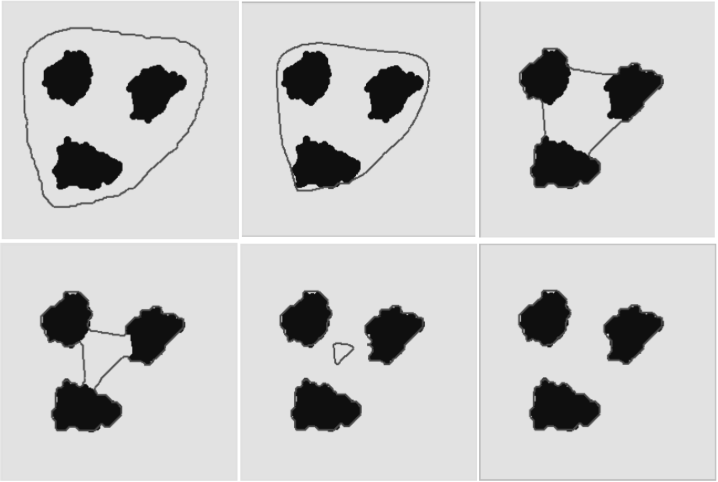
Problem: If ψ is initially a SDF, it may not remain a SDF during evolution.
Solution: Reinitialize periodically (every 50 iterations).

Scheme D: No extension, use $k(x, y)$ within the narrow band, and periodically reinitialize.

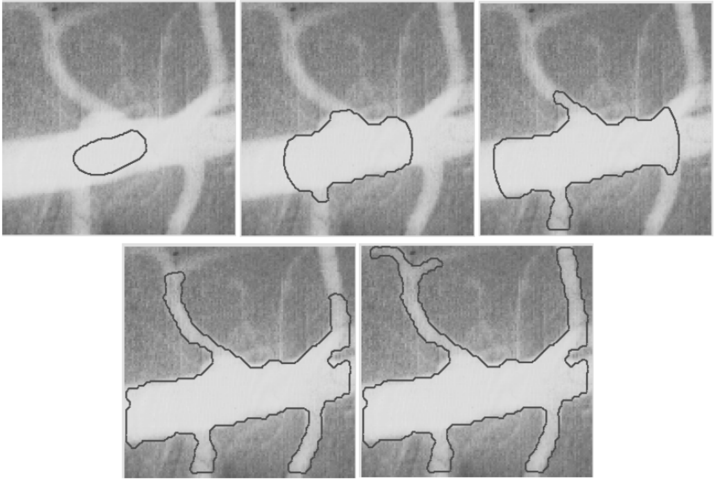
In practice, scheme D performs well and is much faster than the other schemes.

Outline

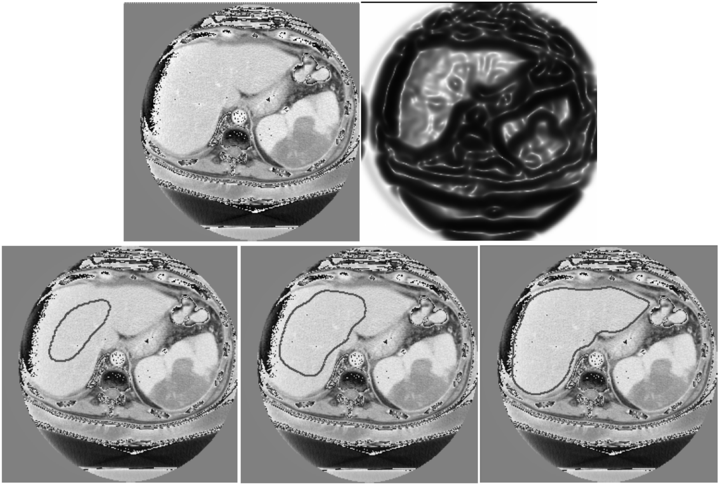
Synthetic Data



Angiogram

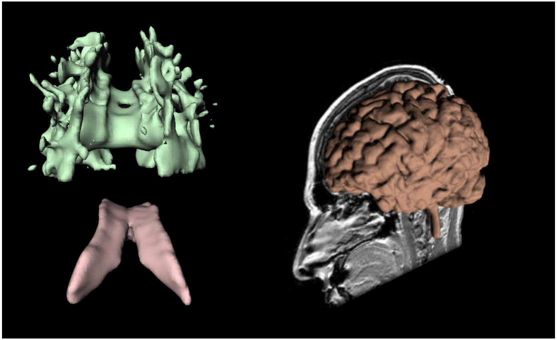


Abdominal CT



In three dimensions

- Level set is the **surface** $\psi(x, y, z) = 0$.
- Compute gradients in 3 dimensions.



Outline

Conclusions

- Shape boundary is able to split and merge to reflect the underlying object geometry.
- When using the narrow band update : Complexity is comparable to snakes.

Problems:

- Speed function may never equal zero, so the front may never reach equilibrium.
- Sensitive to value of the inflation parameter.

Geodesic Active Contours

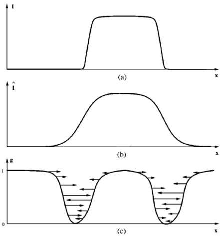
Paper by Caselles, Kimmel, Sapiro is in supplementary reading.

Minimize the weighted length of c(s):

$$\int_{\Omega} k(I(c(s)))||c'(s)||ds$$

The evolution equation is

$$\frac{\partial c}{\partial t} = k(I)\kappa \mathbf{N} - (\nabla k \cdot \mathbf{N})\mathbf{N}$$



Geodesic Active Contours

The curve evolution

$$\frac{\partial c}{\partial t} = k(I)\kappa \mathbf{N} - (\nabla k \cdot \mathbf{N})\mathbf{N}$$

leads to the level set equation

$$\frac{\partial \psi}{\partial t} = -k(I)\kappa ||\nabla \psi|| + \nabla k \cdot \nabla \psi$$

- The first term is the same as the Malladi level set formulation.
- The second term guarantees that the evolution will eventually stop.

Other extensions

Particle level-sets : Eulerian + Lagrangian fluid dynamics.

Douglas Enright, Ronald Fedkiw, Joel Ferziger, Ian Mitchell, ”A Hybrid Particle Level Set Method for Improved Interface Capturing”, Journal of Computational Physics, 2002.

Other extensions

Multi-phase levelsets : Segment more than 2 regions.

For 2 levelset functions ψ^1 and ψ^2 we have

- Region 1 : $\psi^1 < 0$ and $\psi^2 < 0$
- Region 2 : $\psi^1 > 0$ and $\psi^2 < 0$
- Region 3 : $\psi^1 < 0$ and $\psi^2 > 0$
- Region 4 : $\psi^1 > 0$ and $\psi^2 > 0$

In general, n levelset functions can represent 2^n regions.

Next Time

"Active Contours Without Edges."